Elastic space

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The problem

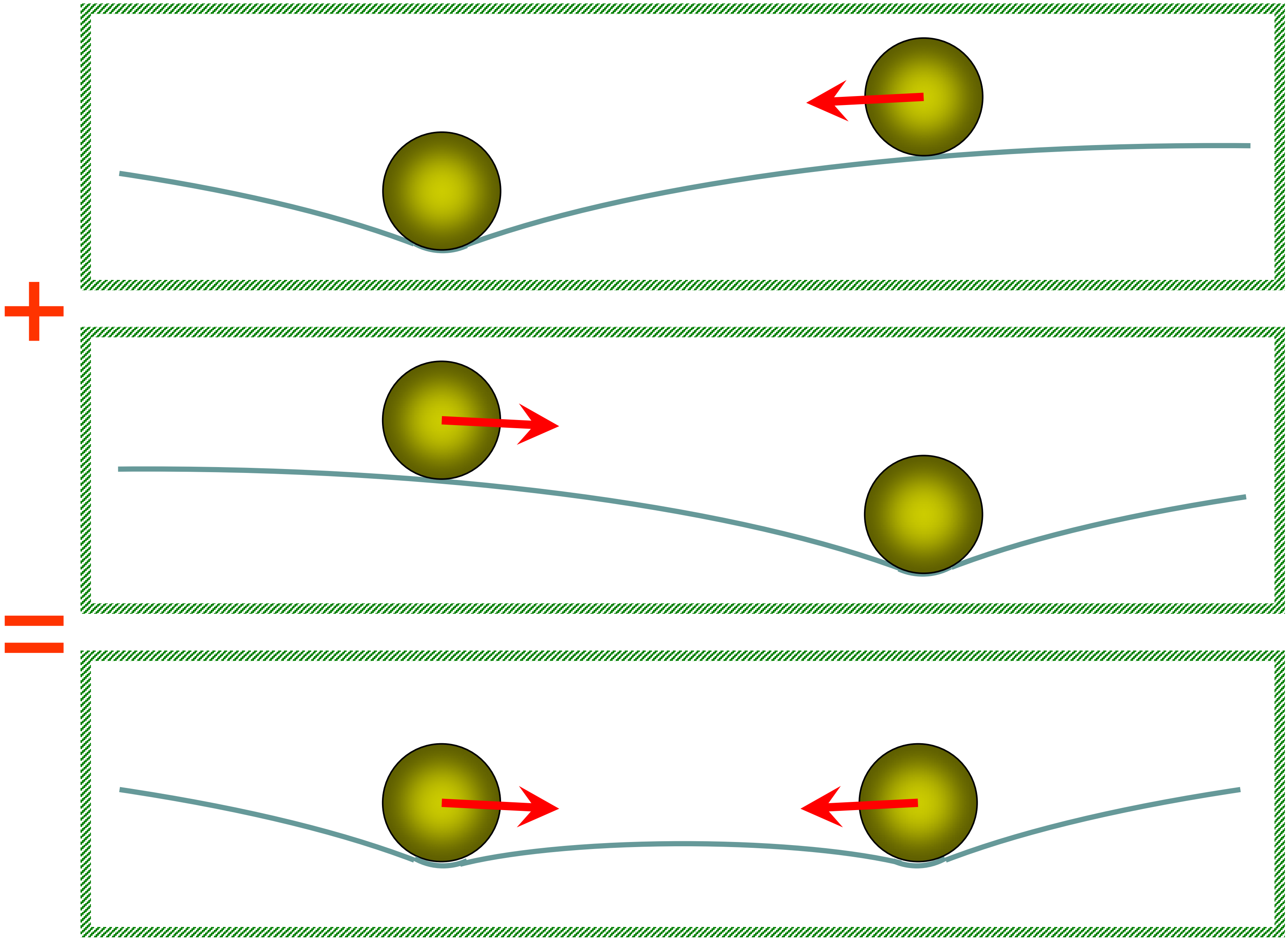
The dynamics and apparent interactions of massive balls rolling on a stretched horizontal membrane are often used to illustrate gravitation. Investigate the system further. Is it possible to define and measure the apparent "gravitational constant" in such a "world"?

Membrane deflection and attractive force

Membrane tension

- Membrane tension σ is constant for all directions.
- Membrane tension σ is high enough, so a membrane inclination angle α is small everywhere.

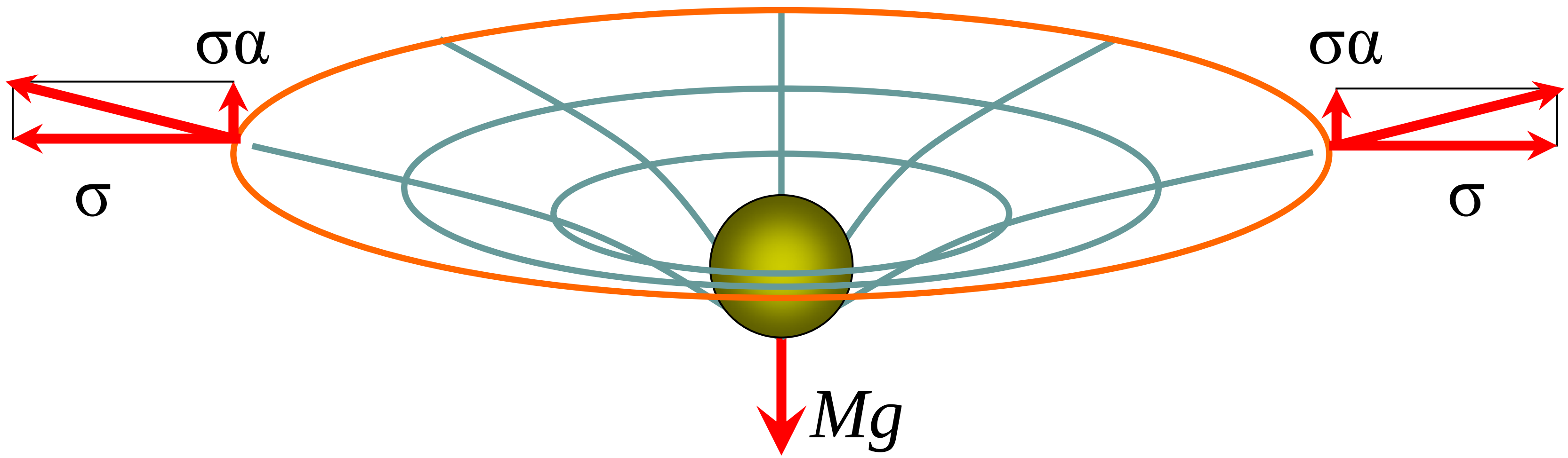
Superposition principle



Membrane deflection

σ — membrane tension

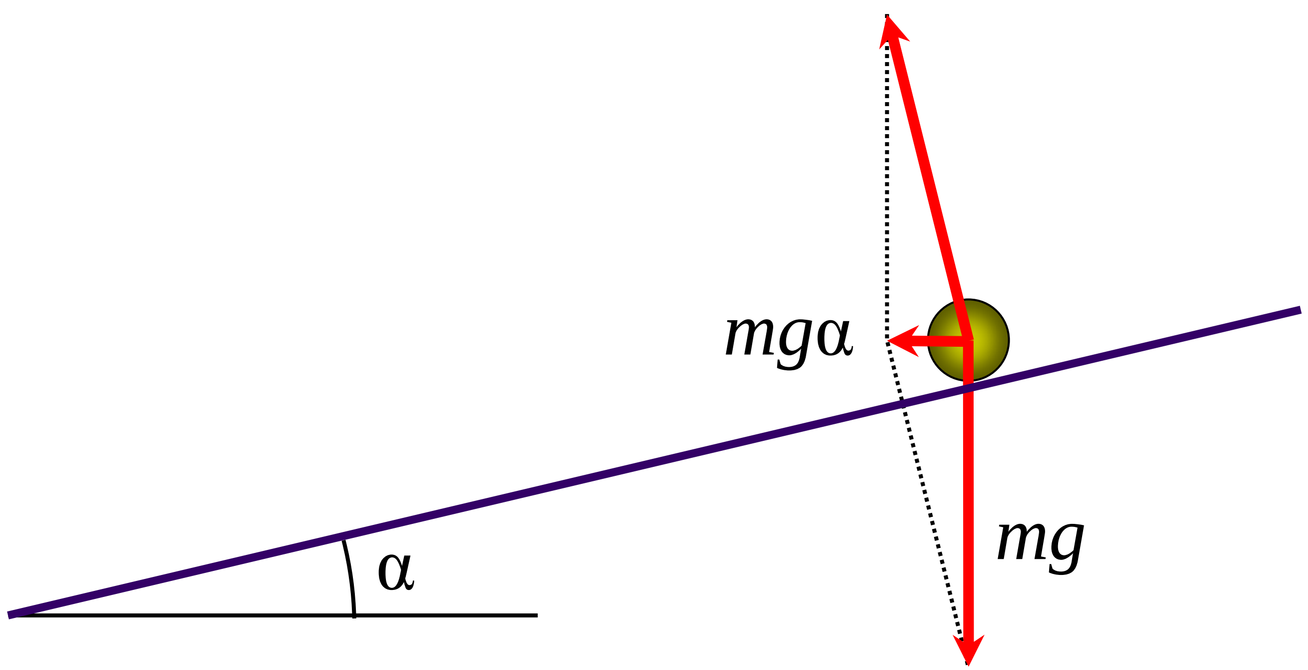
$\alpha \ll 1$ — membrane inclination angle



$$Mg = 2\pi r \cdot \sigma\alpha \quad \Rightarrow \quad \alpha = \frac{Mg}{2\pi r \sigma}$$

Horizontal acting force

- Horizontal force acting on the ball is equal to $mg\alpha$, where α is the membrane inclination angle in the absence of the ball.



“Law of universal gravitation”

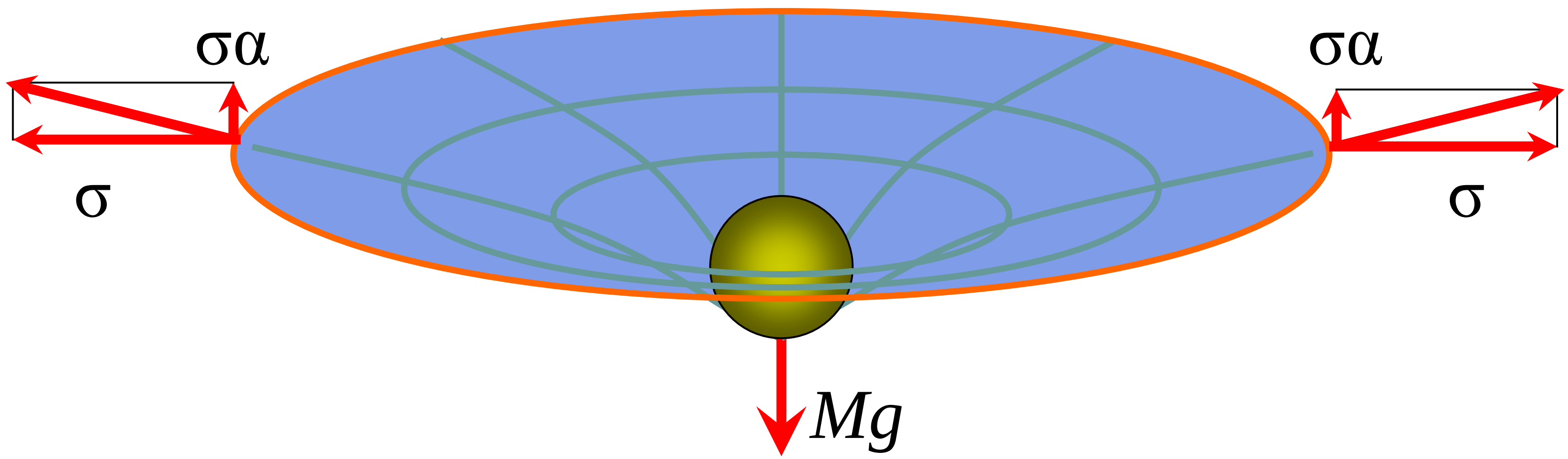
Two balls lying on the membrane attract each other. Their attractive force is:

$$F_{12} = \frac{g^2}{2\pi\sigma} \cdot \frac{M_1 M_2}{r}$$

Gravitational constant

Instead r^2 in 3-D case

The weight of membrane



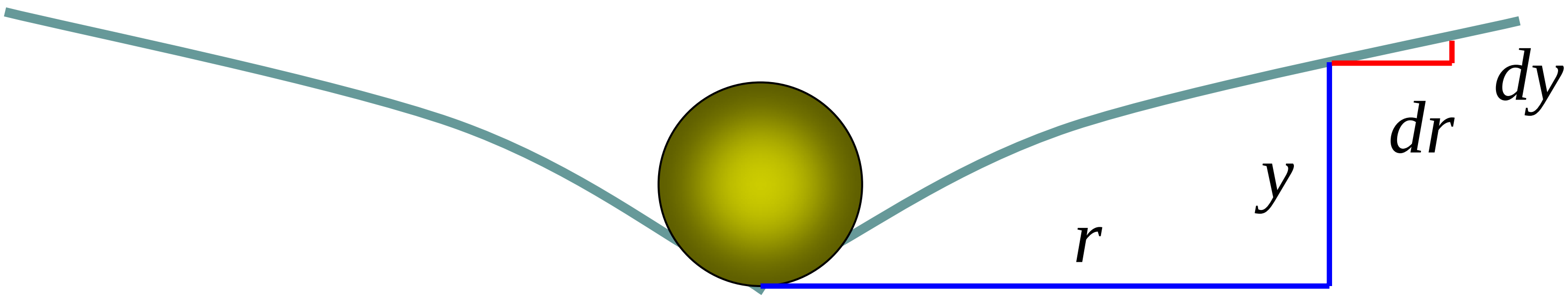
Balance condition:

$$(M + \pi r^2 \rho) g = 2\pi r \sigma \alpha$$

Membrane weight is negligible, if

$$\pi r^2 \rho \ll M$$

Membrane profile

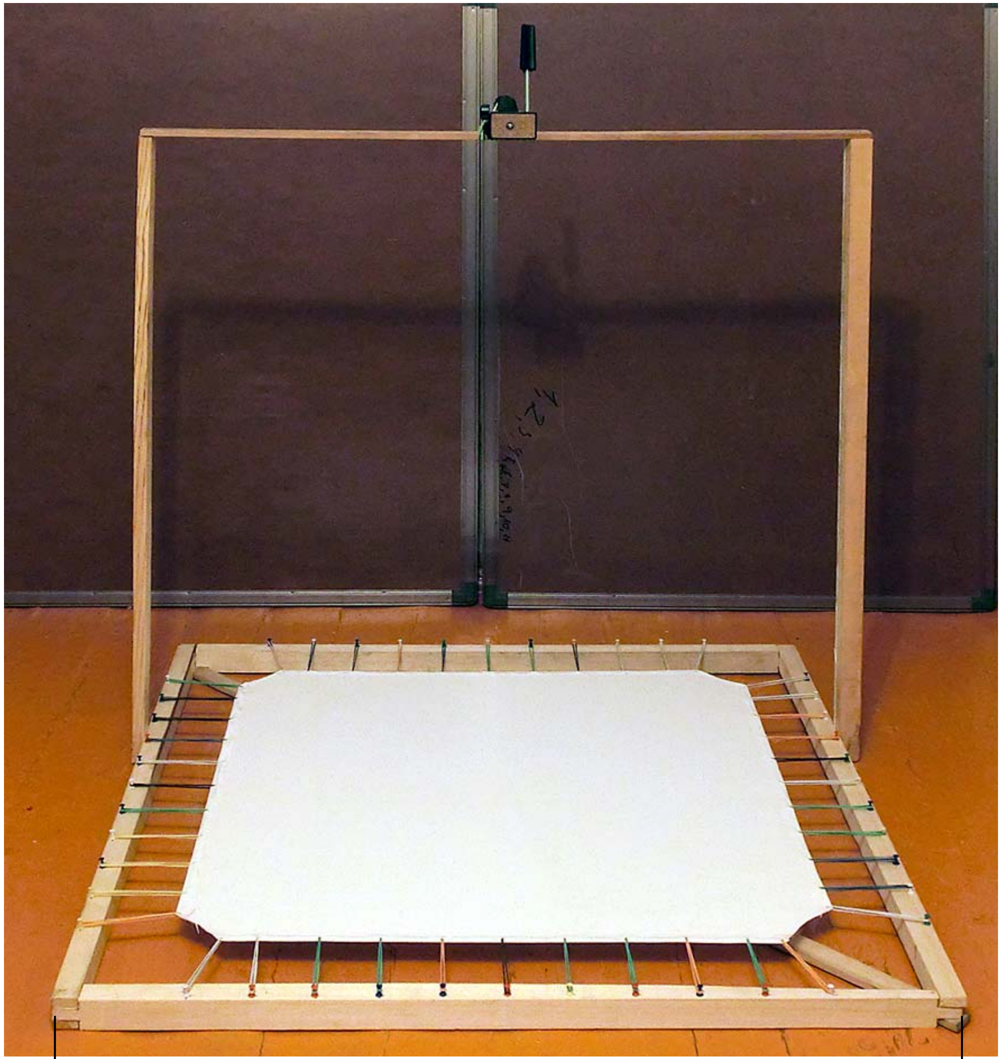


Profile equation in the small angle approximation :

$$\frac{dy}{dr} = \tan\alpha \approx \alpha = \frac{Mg}{2\pi\sigma} \cdot \frac{1}{r}$$

$$y(r) = \frac{Mg}{2\pi\sigma} \cdot \int_{r_0}^{r} \frac{dr}{r} = \frac{Mg}{2\pi\sigma} \cdot \ln\left(\frac{r}{r_0}\right)$$

Experimental setup

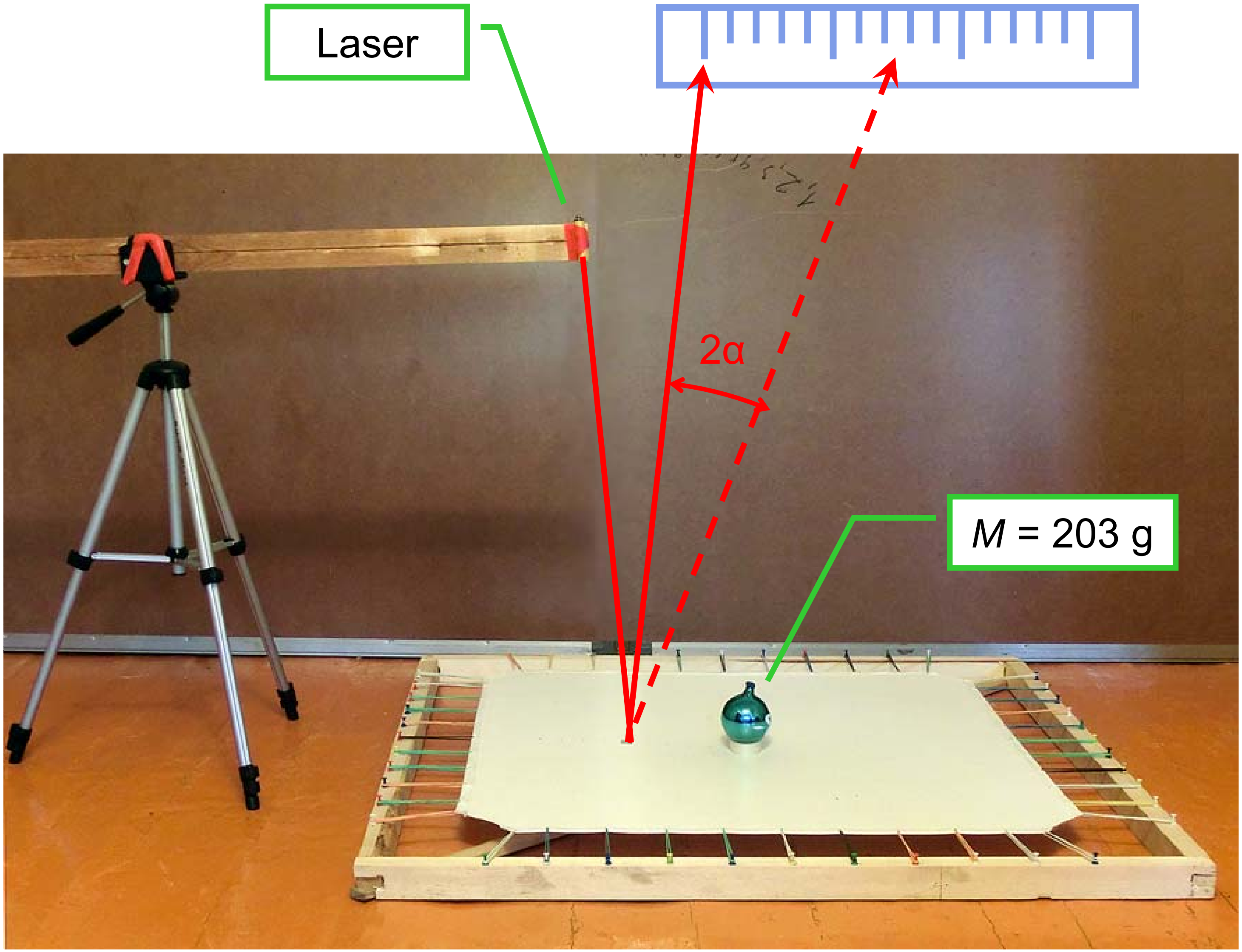


95 cm

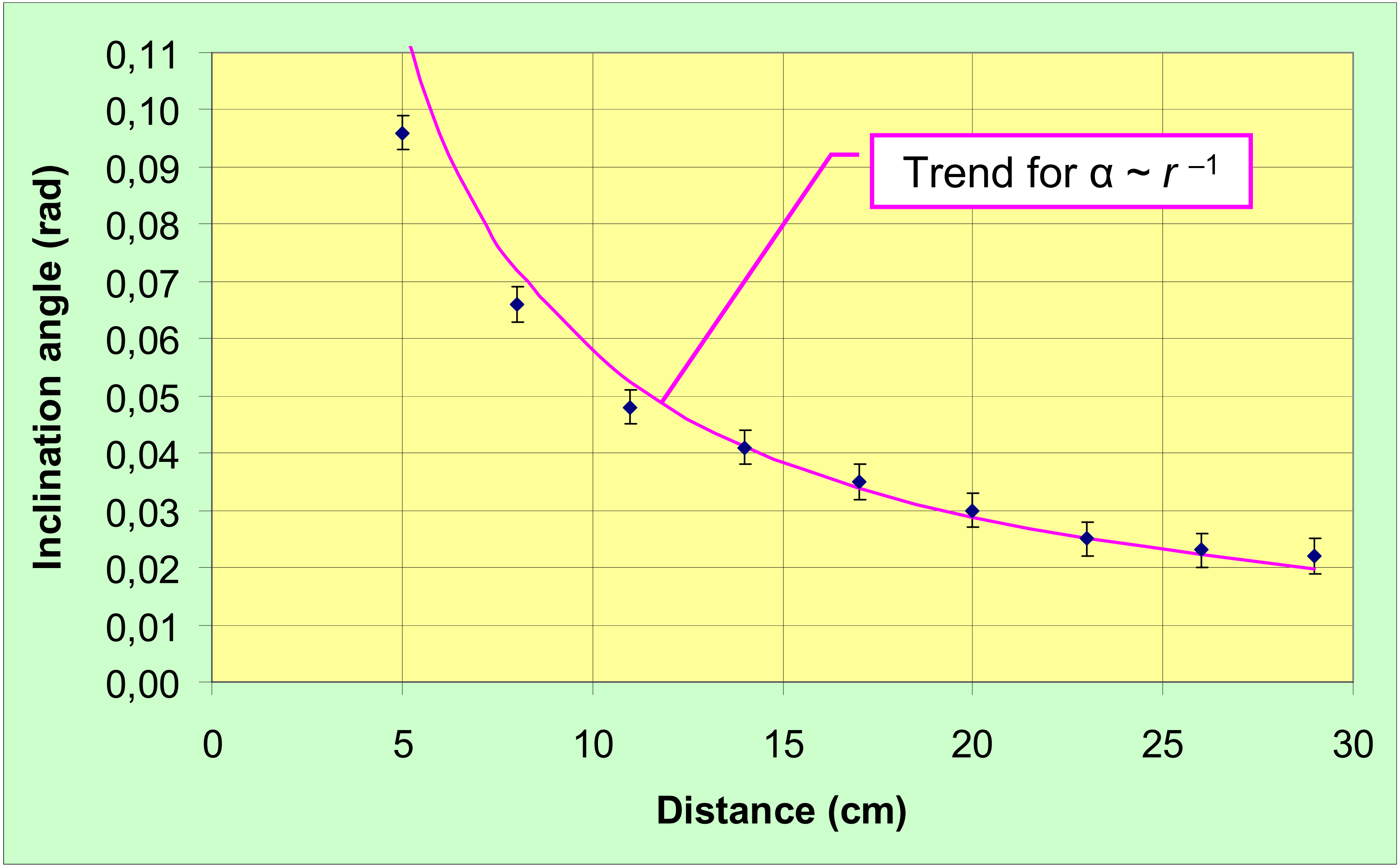
Membrane properties

- Density **ρ = 240 g/m^2.**
- Membrane tension **σ = 70 N/m** is defined by the elongation of rubber extensions.
- In fact, this tension is less because of hard slats mounted at the rim of membrane.

Measurement of inclination angle



Experimental diagram



Membrane properties

- Membrane tension

$$\sigma = \frac{mg}{2\pi\alpha r}$$

Calculated value **σ = 55 ± 5 N/m.**

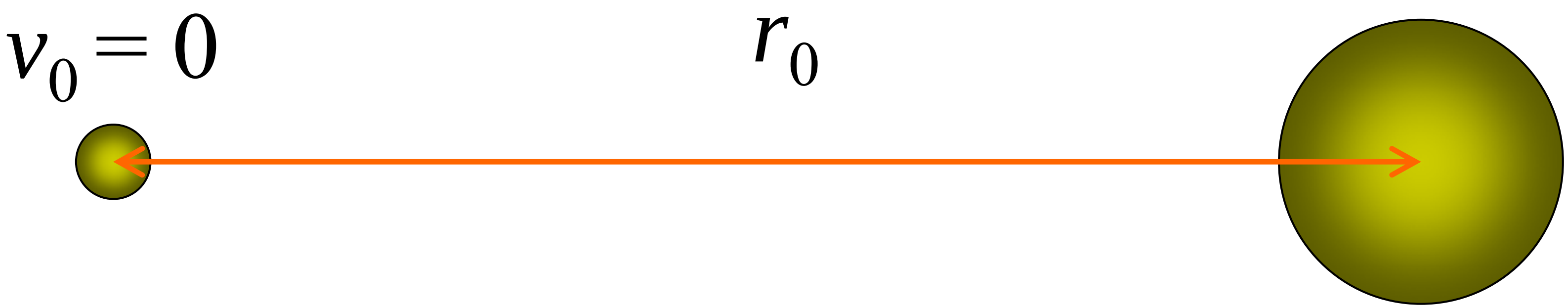
- Gravitational constant

$$G = \frac{g^2}{2\pi\sigma}$$

Calculated value **$G = 0.28 \pm 0.03\ m^2/(kg \cdot s^2)$.**

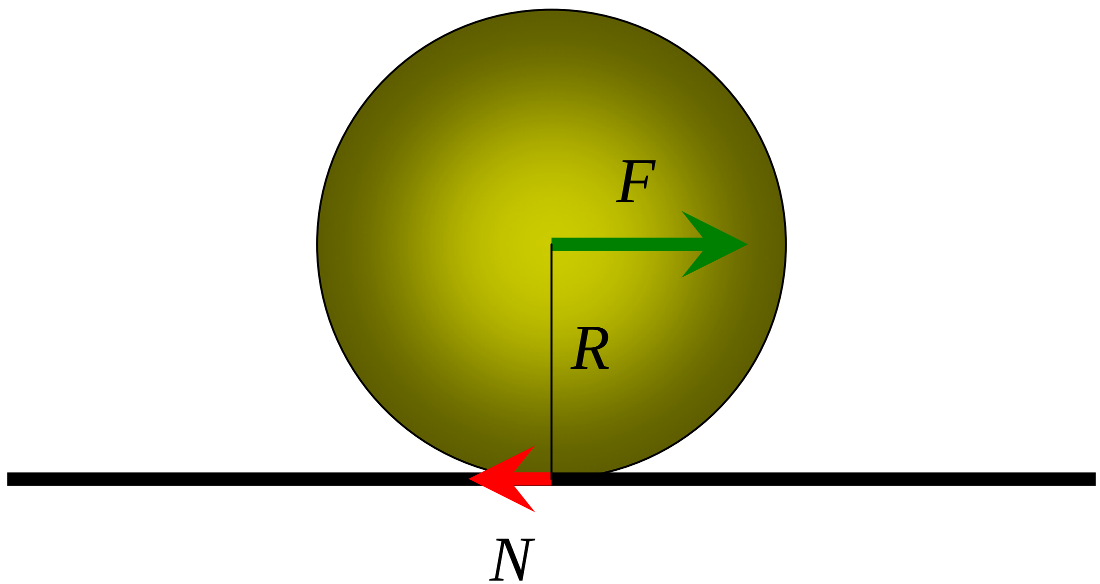
Radial motion

Statement of the problem



The ball rolls radially
with zero initial velocity

Effective mass



$$\begin{cases} m\dot{v} = F - N \\ I\dot{\omega} = NR \\ v = \omega R \end{cases} \Rightarrow \left(m + \frac{I}{R^2}\right)\dot{v} = F$$

Effective mass

Theoretical calculation

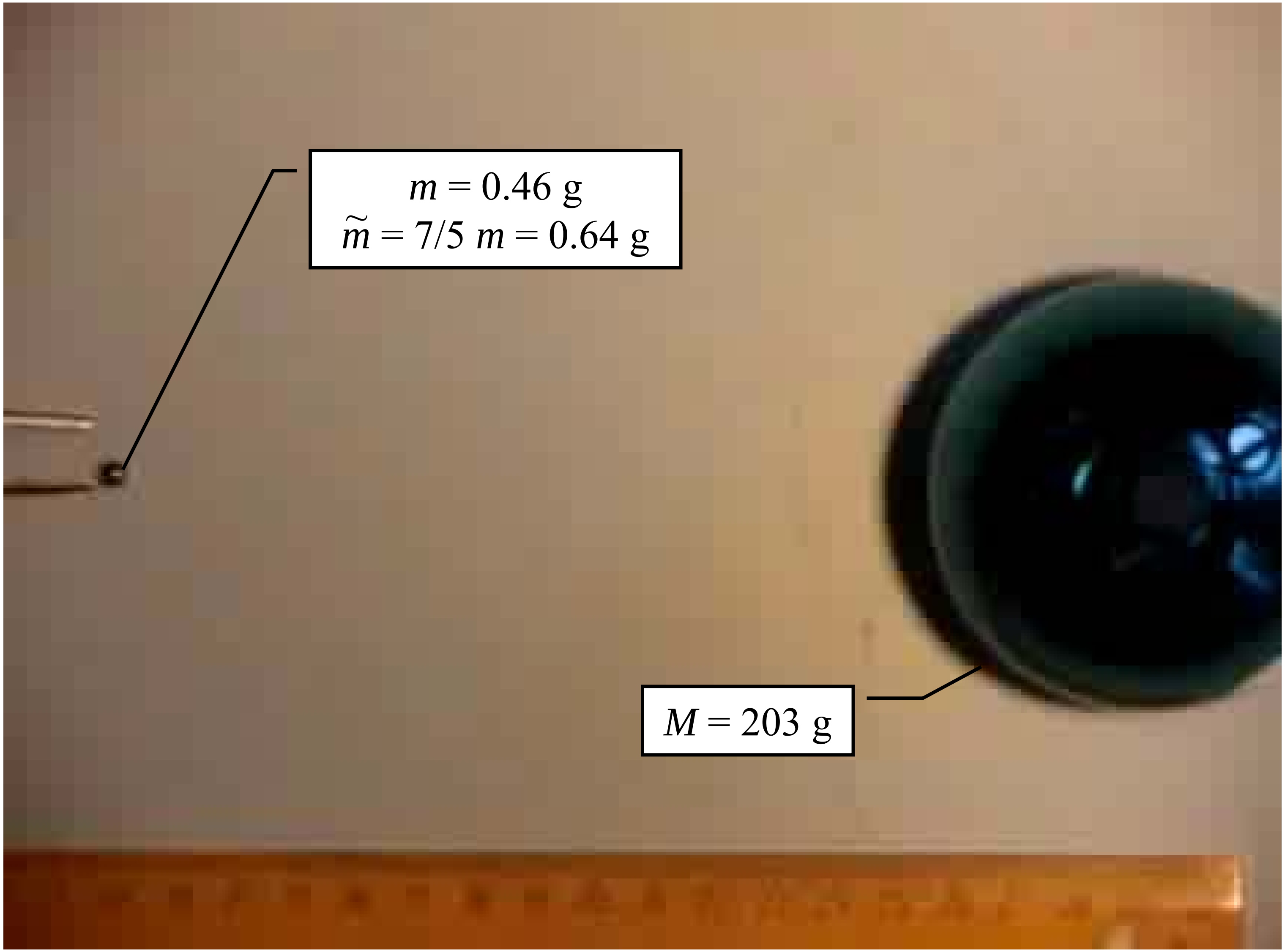
Energy conservation law (with $v(r_0) = 0$):

Effective mass

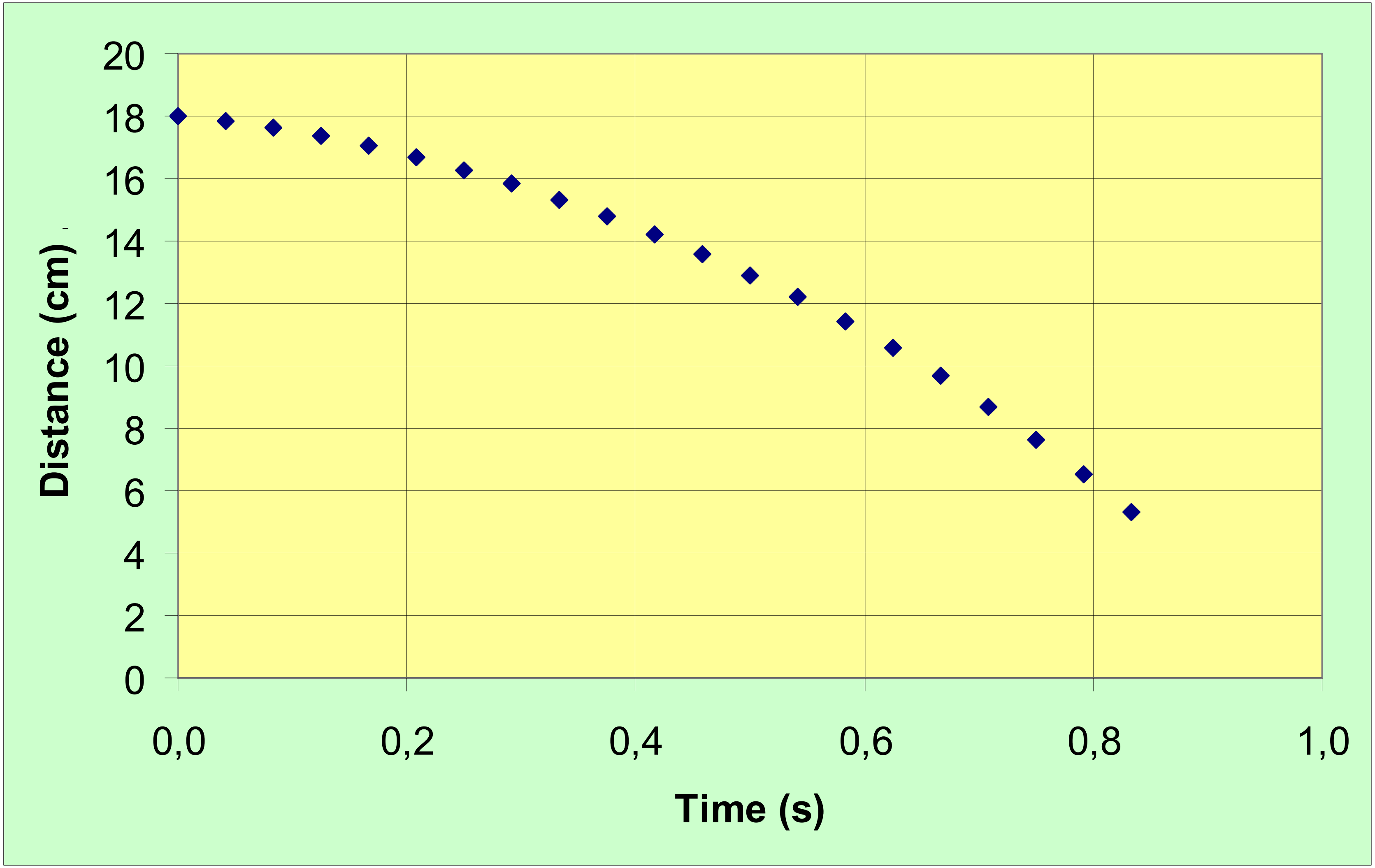
$$\frac{\tilde{m} v^2}{2} + GmM \cdot \ln\left(\frac{r}{r_0}\right) = 0$$

$$v(r) = \sqrt{2MG\frac{m}{\tilde{m}} \cdot \ln\left(\frac{r_0}{r}\right)}$$

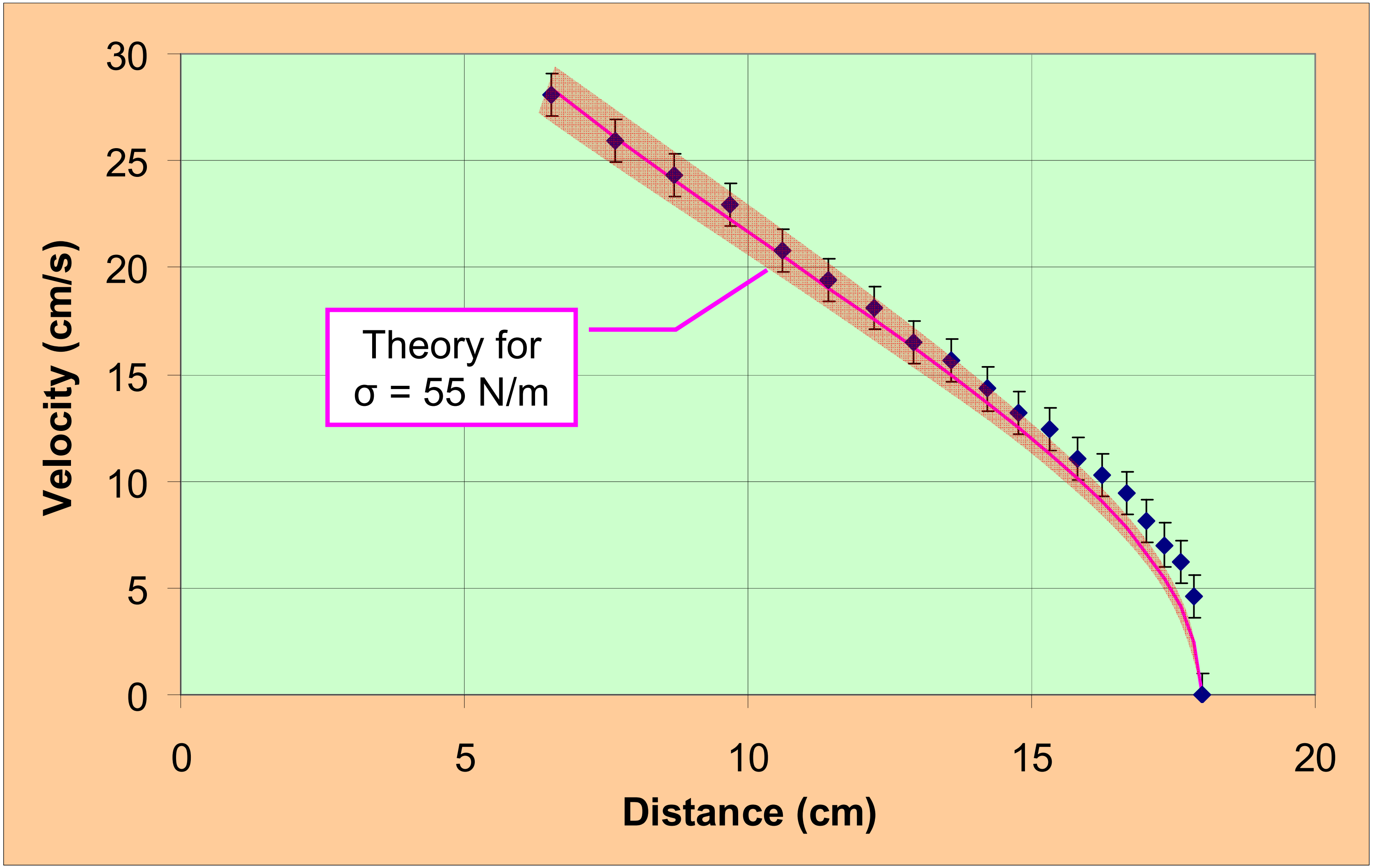
Experiment (video 240 fps)



Distance vs. time

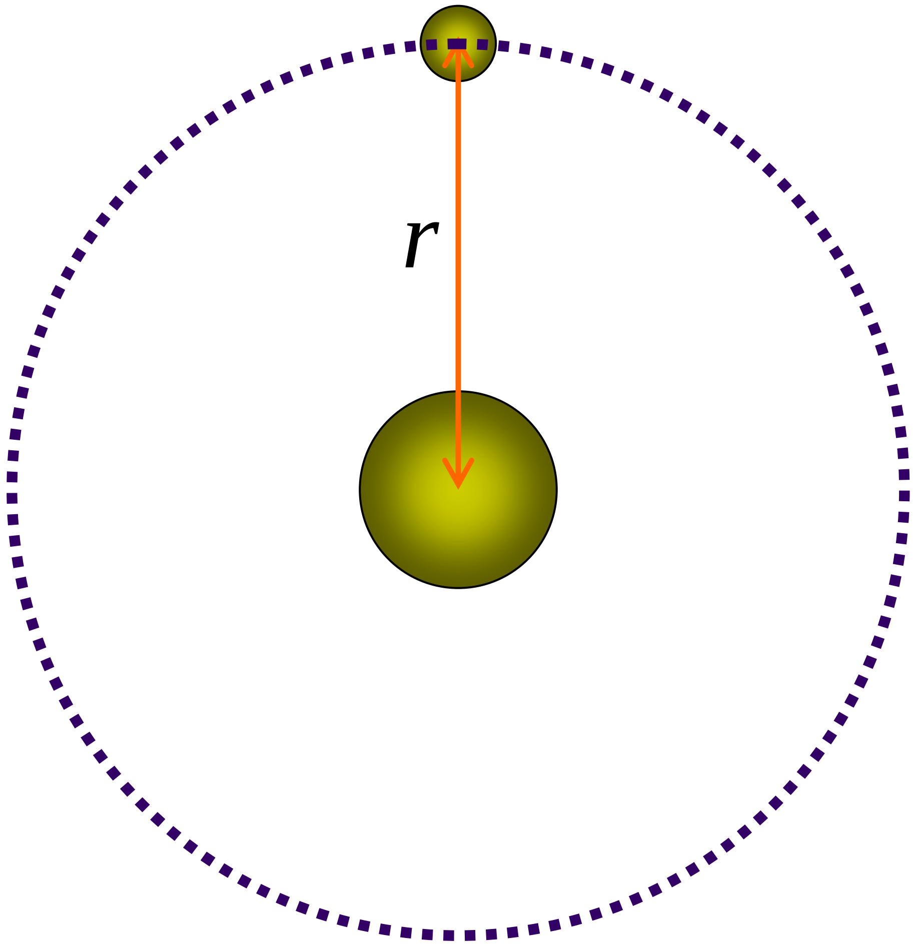


Velocity vs. distance



Circular motion

Statement of the problem



Small ball rolls on a circular orbit.
Big ball is held in place by friction.

Theory

Second Newton's law:

$$G \cdot \frac{Mm}{r} = \frac{\tilde{m} v^2}{r}$$

«Gravitation force»

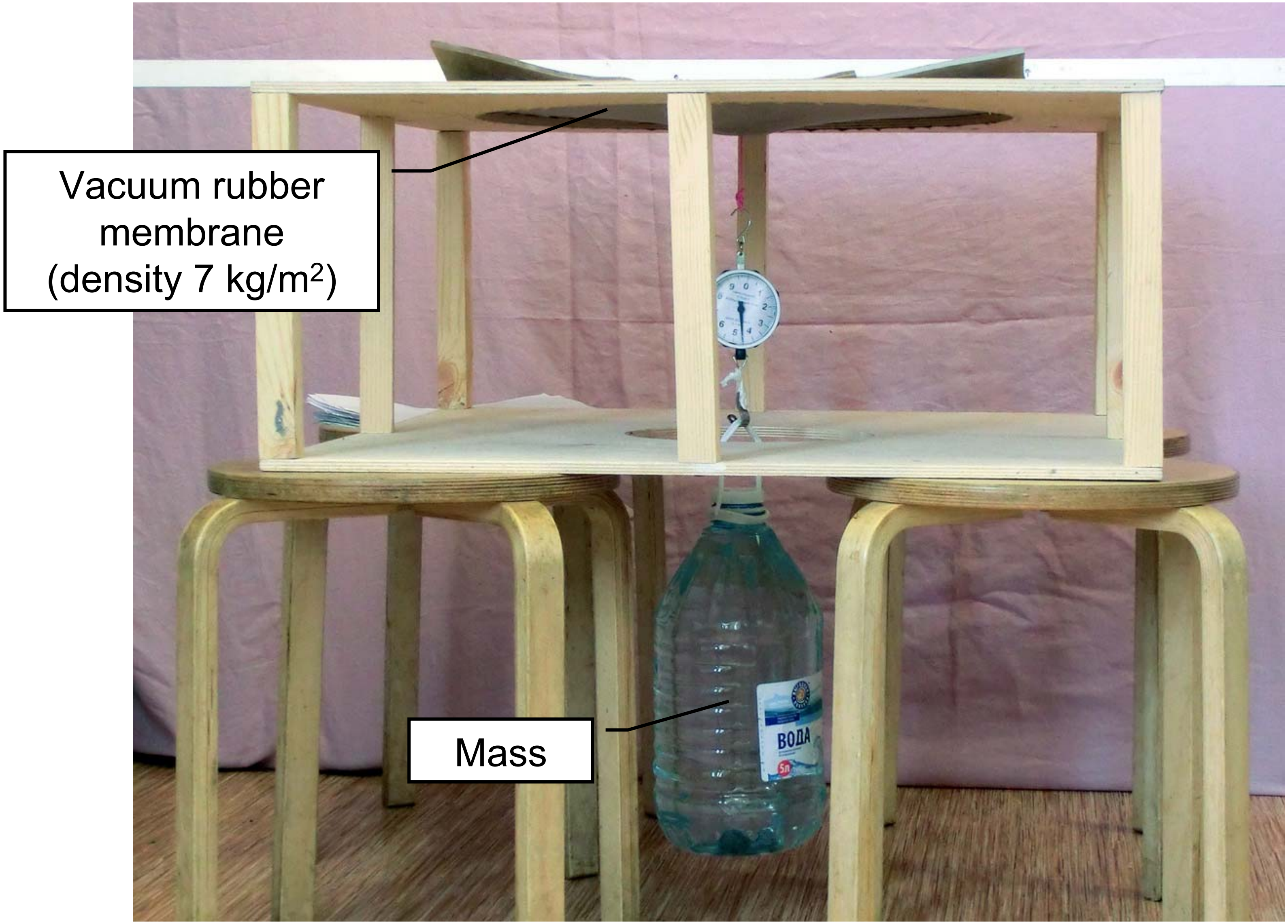
Effective mass

Centripetal acceleration

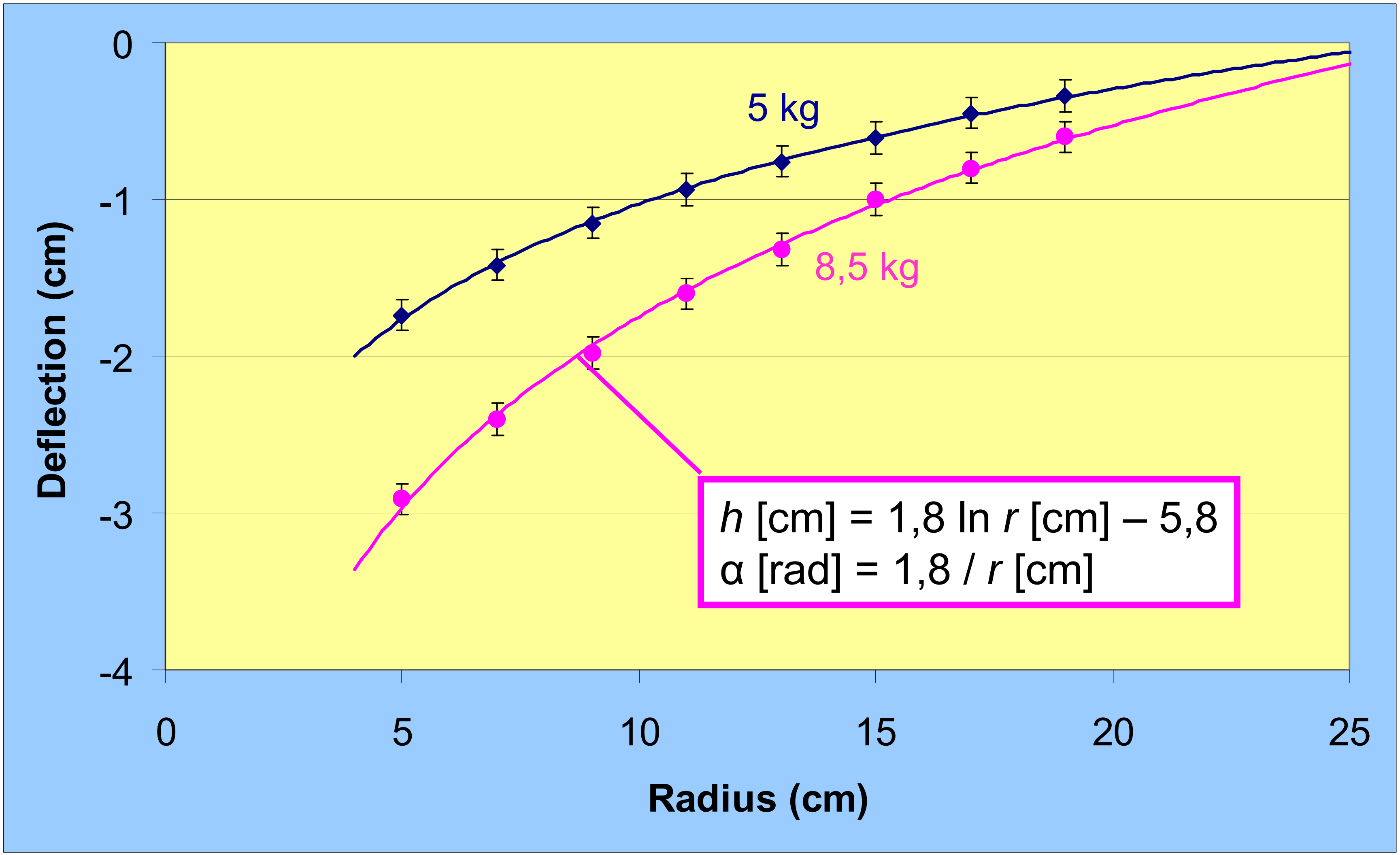
$$v = \sqrt{GM \frac{m}{\tilde{m}}}$$

Velocity of the ball on a circular orbit is independent of the radius.

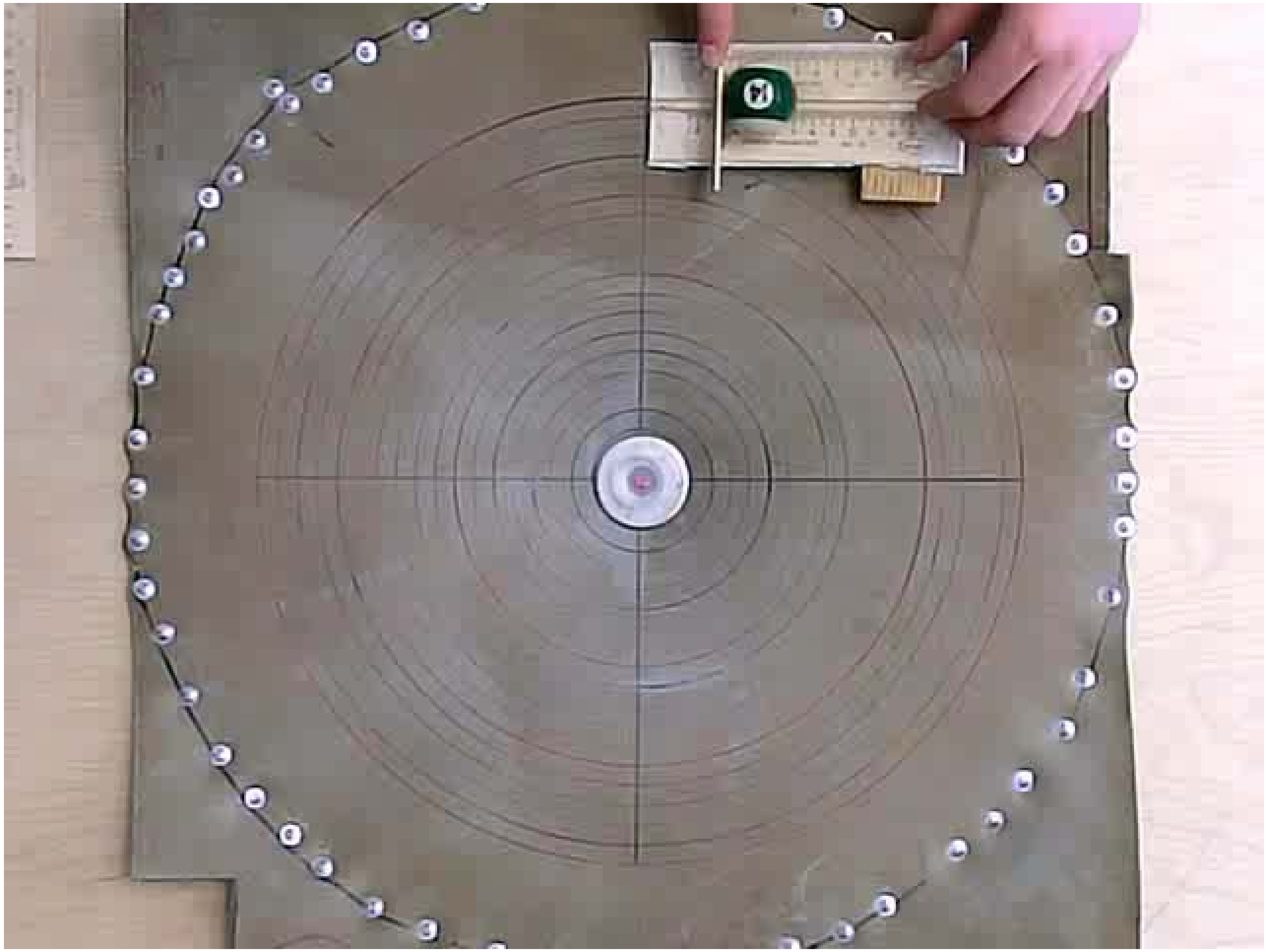
Experimental setup



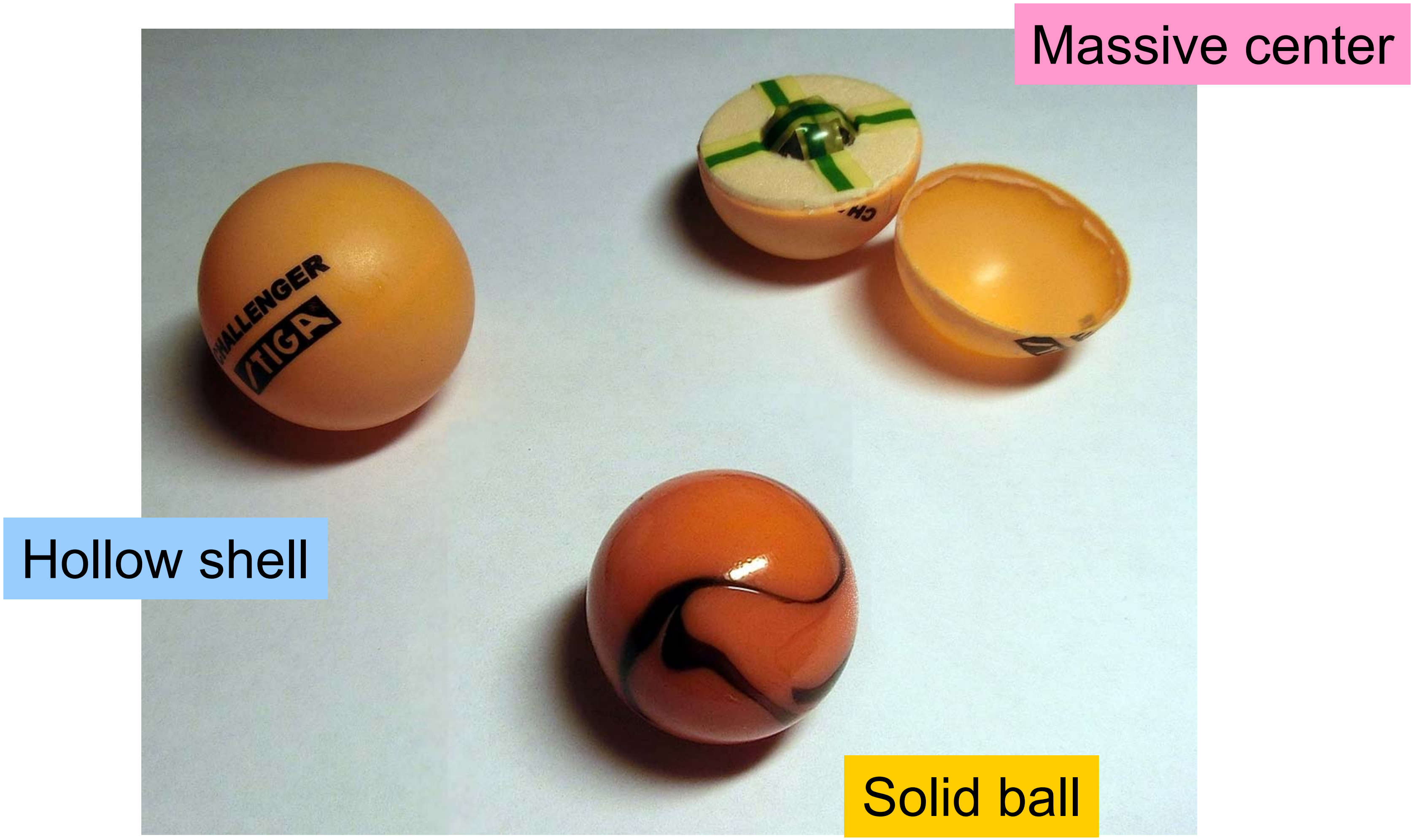
Membrane profile and inclination



Rolling on a circular orbit (video)



Used balls



Comparison of theory with experiment

Ball	*m*/m*	Theoretical velocity (cm/s)	Experimental velocity (cm/s)
Massive center	1	42.5	42.5 ± 1
Solid ball	7/5	35.7	35 ± 1
Hollow shell	5/3	32.7	32 ± 1

General shape of the orbit

Orbital equation

Angular momentum conservation law :

$$J = \tilde{m} r^2 \dot{\varphi} = \mathrm{const}$$

Energy conservation law:

$$E = \frac{\tilde{m}\dot{r}^2}{2} + \frac{\tilde{m} r^2 \dot{\varphi}^2}{2} + U(r) = \mathrm{const}$$

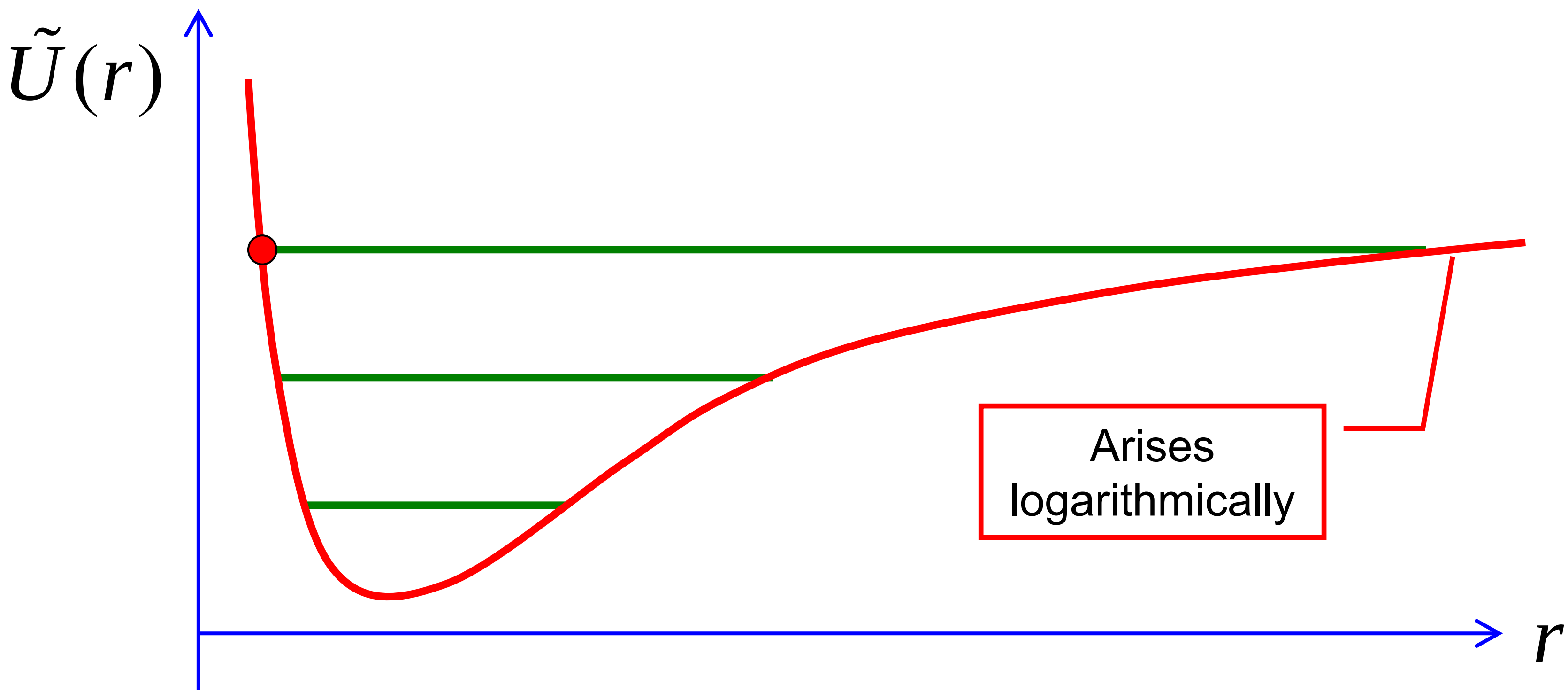
Orbital equation:

$$d\varphi = \frac{J \cdot dr}{r^2 \cdot \sqrt{2\tilde{m}(E - U(r)) - \frac{J^2}{r^2}}}$$

The orbit always is bounded

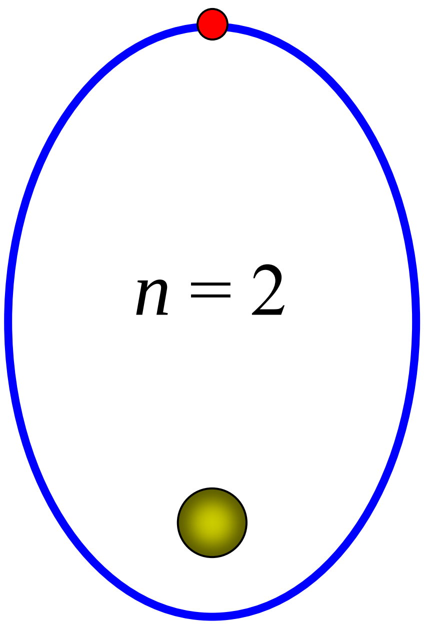
Effective potential for $F \sim 1/r$:

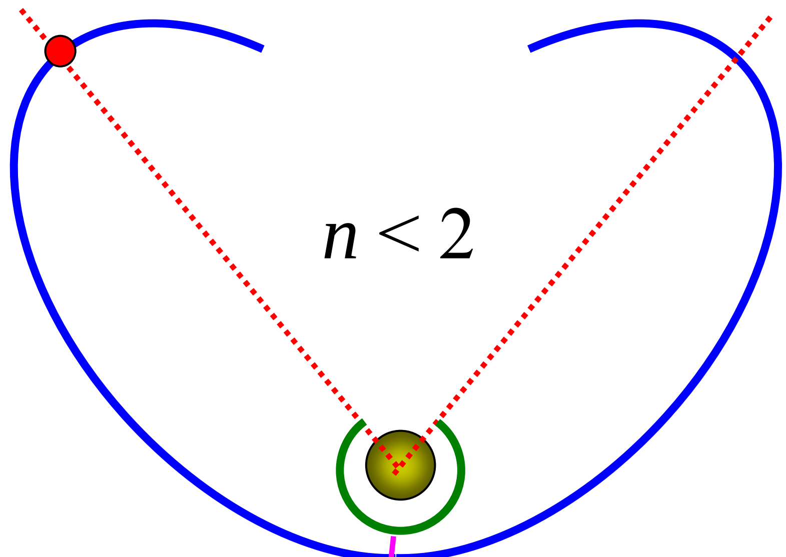
$$\tilde{U}(r) = U(r) + \frac{J^2}{2\tilde{m}r^2} = G \cdot Mm \cdot \ln\left(\frac{r}{r_0}\right) + \frac{J^2}{2\tilde{m}r^2}$$



Rotation of the apsis

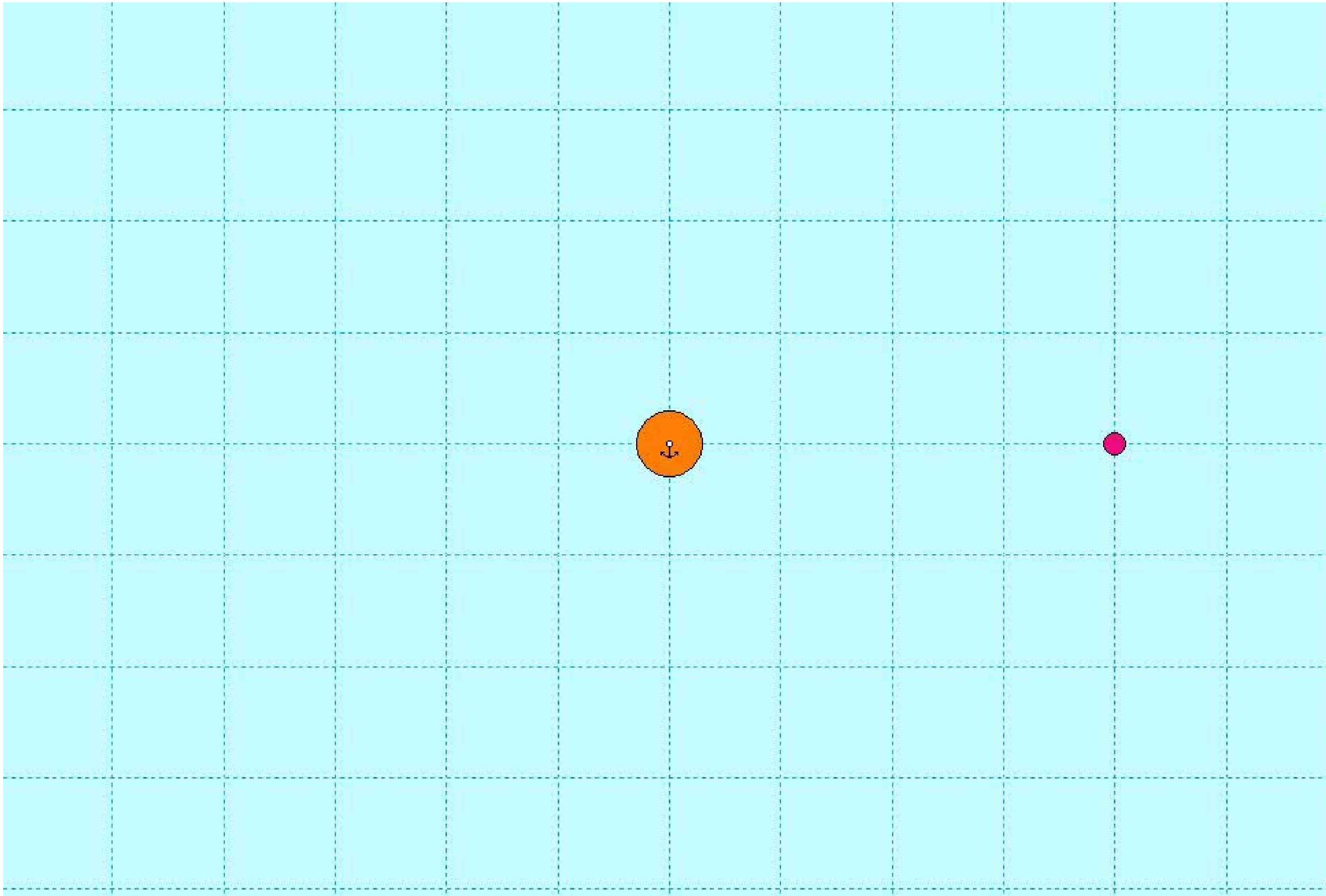
$$F(r) \sim \frac{1}{r^n}$$



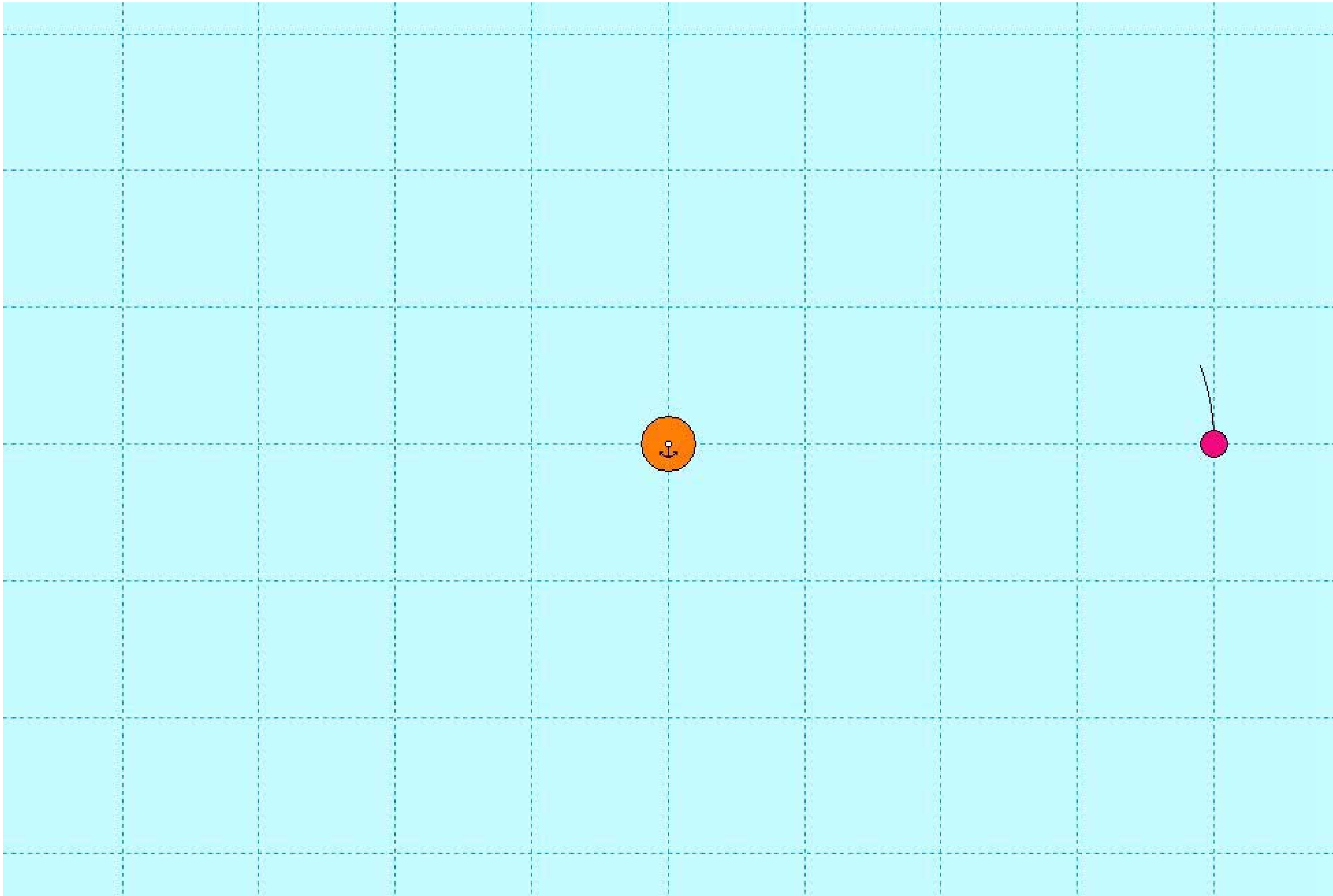


The force increasing towards the centre is not enough to rotate the apsis on 360°

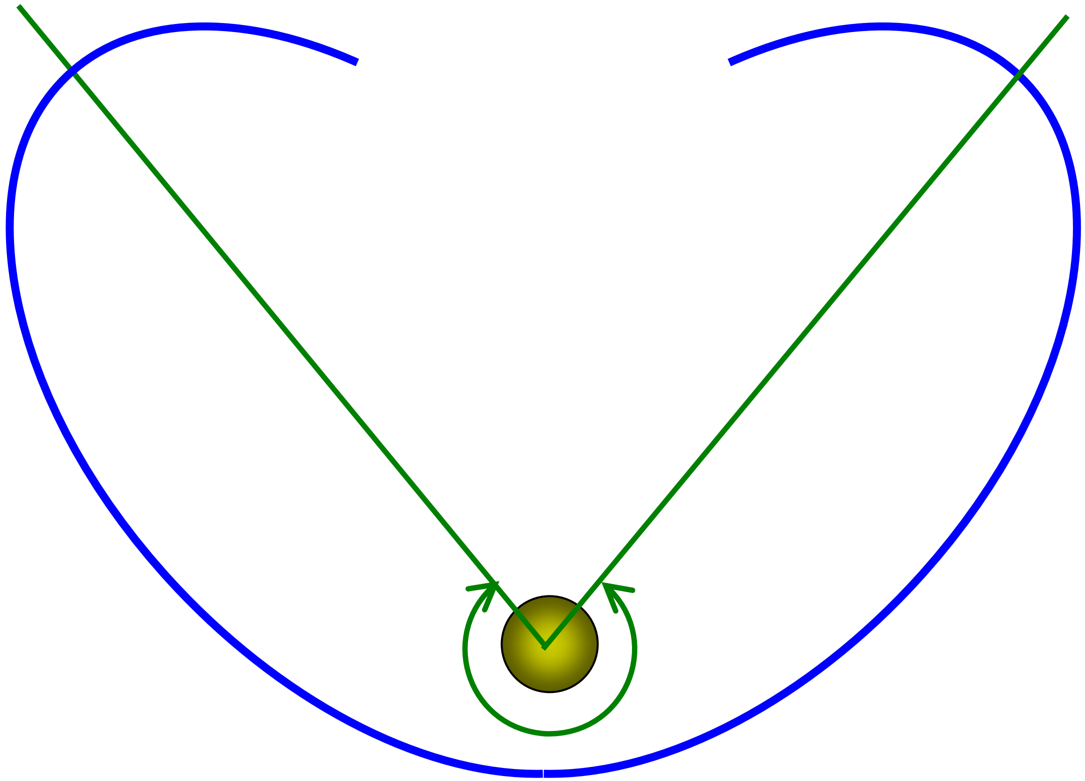
Orbital motion without friction



Orbital motion with friction



Angle between the apsides

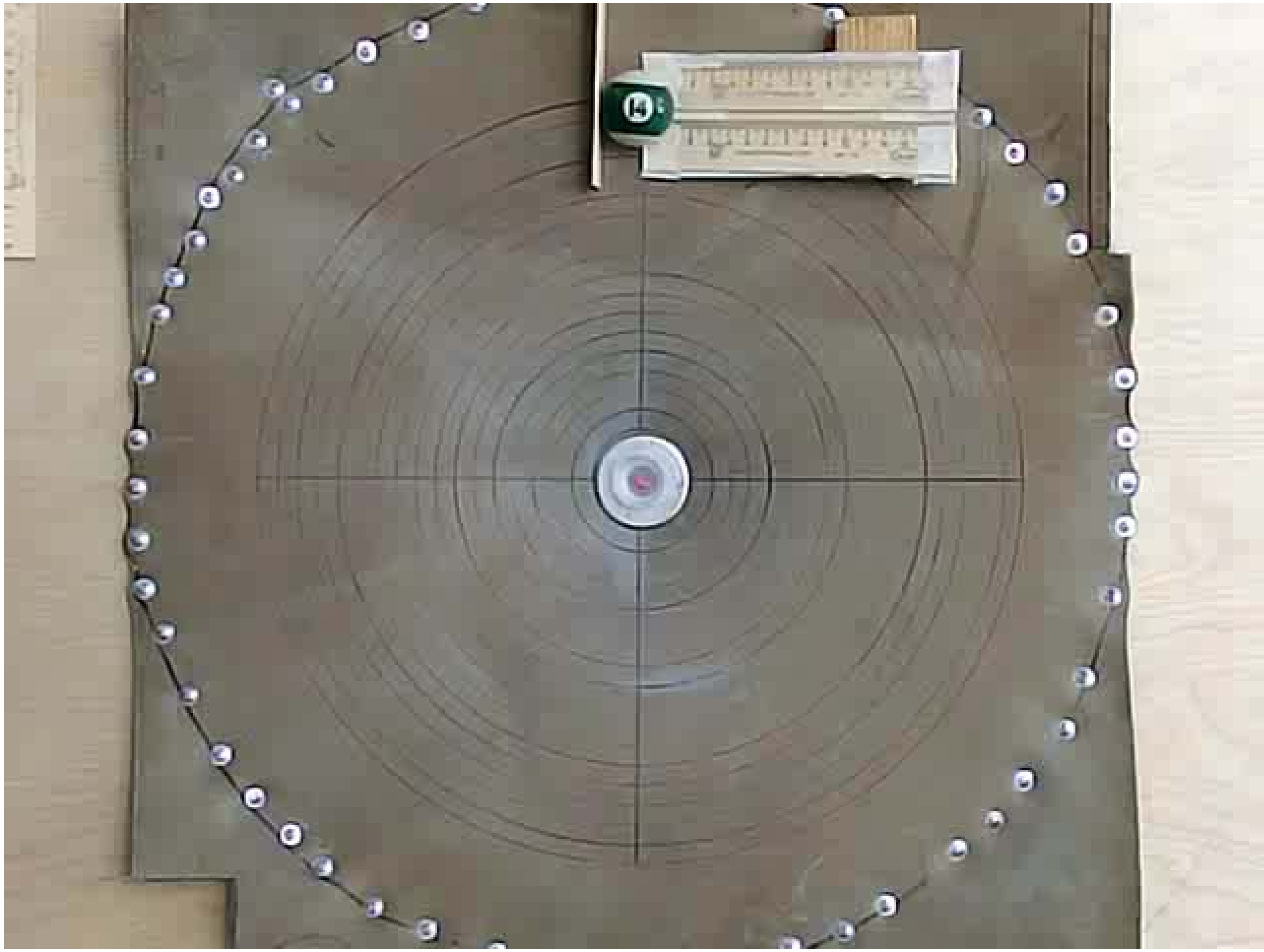


In computer simulation

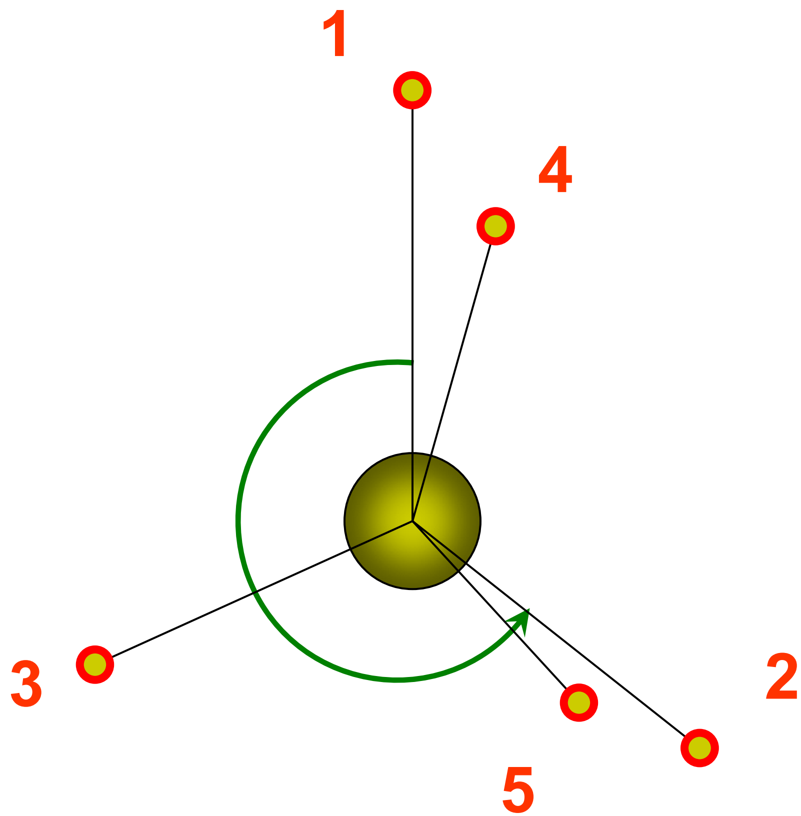
$\alpha = 250° \pm 1°$

Orbital motion (video 1)

Orbital motion (video 2)



Consecutive apsides



In this experiment

$\alpha = 235° \pm 5°$

Two moving balls

Two balls move on spirals

The heavy ball entrains the light ball

Very complex motion

- In the simplest model of this motion both balls are moving in their orbits around their common center of mass, while the center itself moves uniformly.
- Real situation contains two complications:
  - energy loss due to friction;
  - interaction of the balls with “the boundary of the world”.

Summary

Conclusions

- Attractive force between the balls is **inversely proportional to the radius r** instead r^2 in the case of real gravity.
- Kepler's first law (the orbit of every planet is an ellipse) is no longer satisfied.
- Kepler's second law (a line joining a planet and the Sun sweeps out equal areas during equal intervals of time) is still valid.
- Kepler's third law now says that for the circular orbit the period of a planet is proportional to its orbital radius.

References

- Synge J.L. (1960) *Classical dynamics.*
- Borisov A.V. , Mamaev I.S. , Kilin A.A. (2002) "The rolling motion of a ball on a surface. New integral and hierarchy of dynamics". *Regular and chaotic dynamics,* **7**, 201–219.

Thank you for your attention!