Soliton

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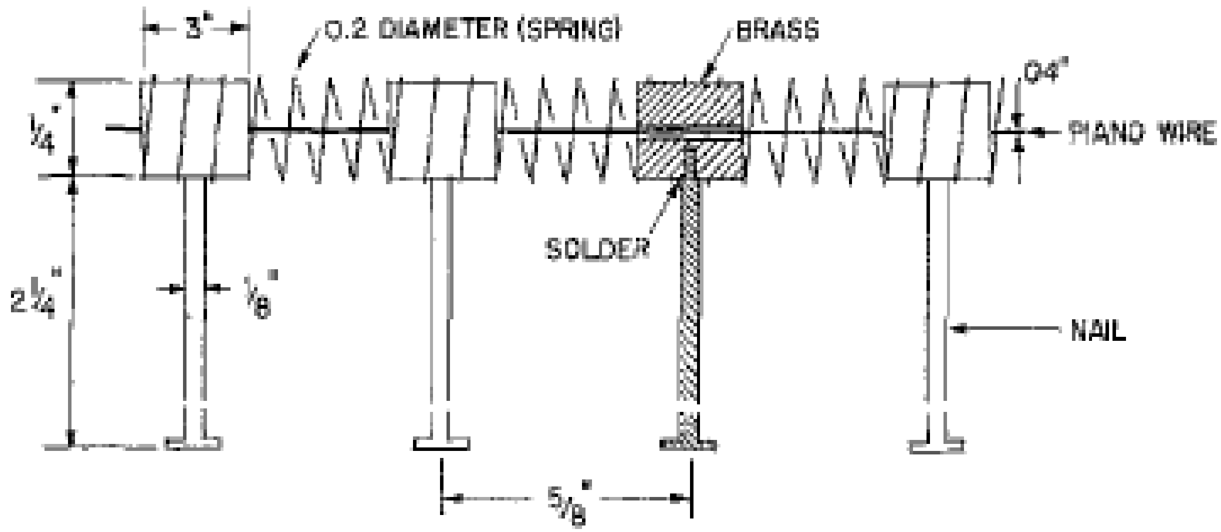
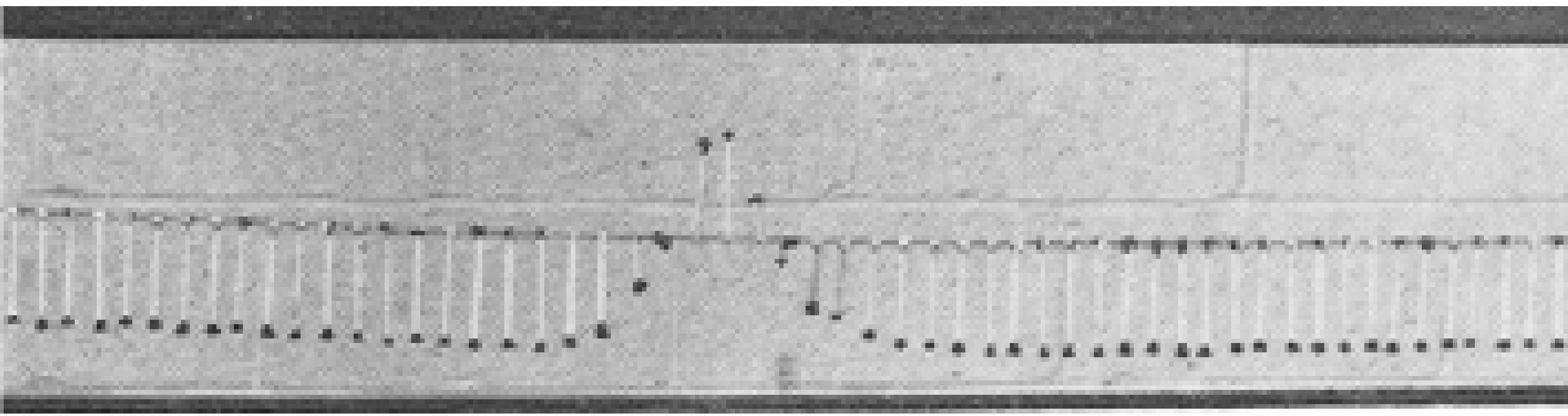
The problem

A chain of similar pendula is mounted equidistantly along a horizontal axis, with adjacent pendula being connected with light strings. Each pendulum can rotate about the axis but can not move sideways. Investigate the propagation of a deflection along such a chain. What is the speed for a solitary wave, when each pendulum undergoes an entire 360º revolution?

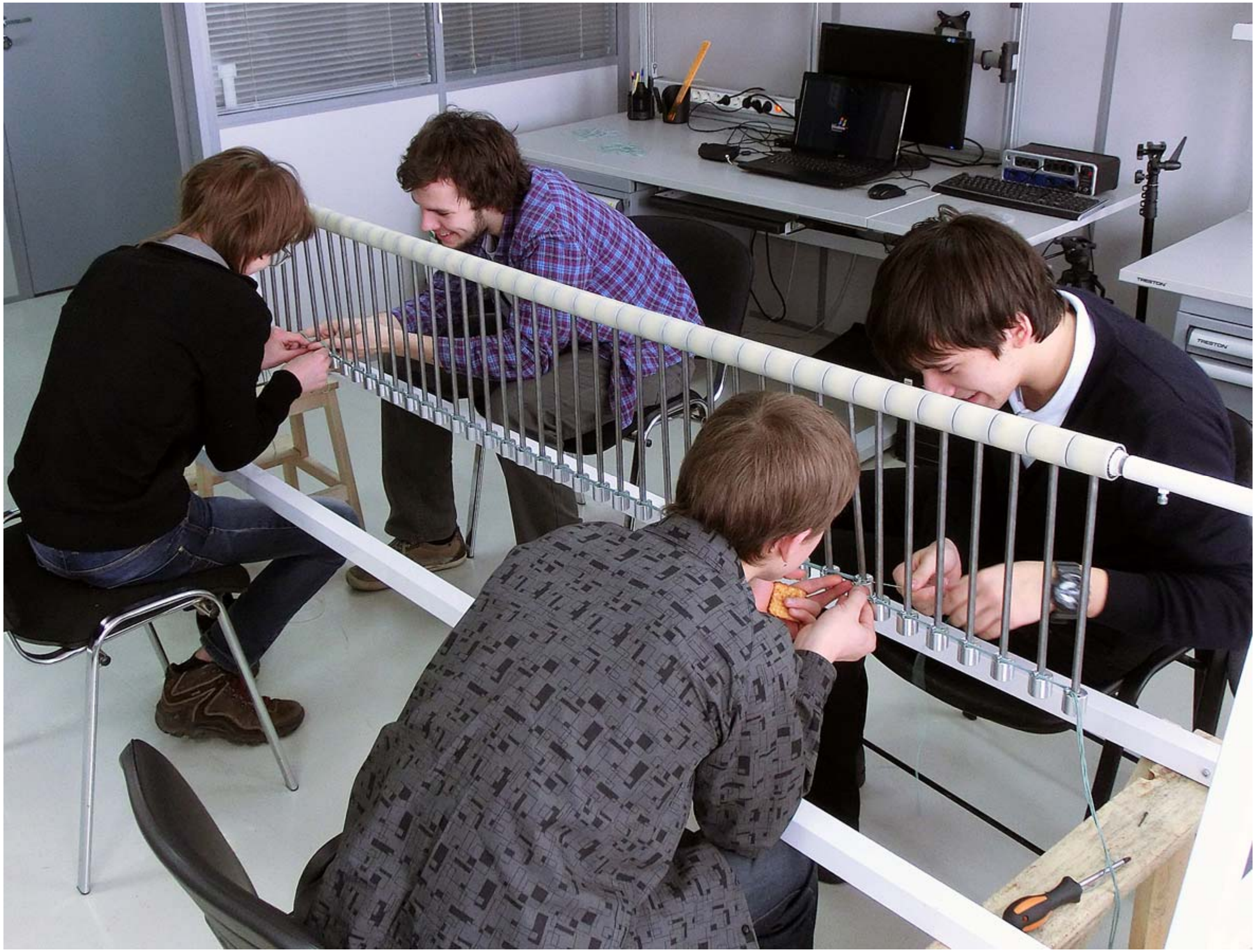
Previous investigations





Scott A. C. (**1969**) “A non-linear Klein-Gordon equation”. *Am. J. Phys.*, **37**, 52–61.

Experimental setup



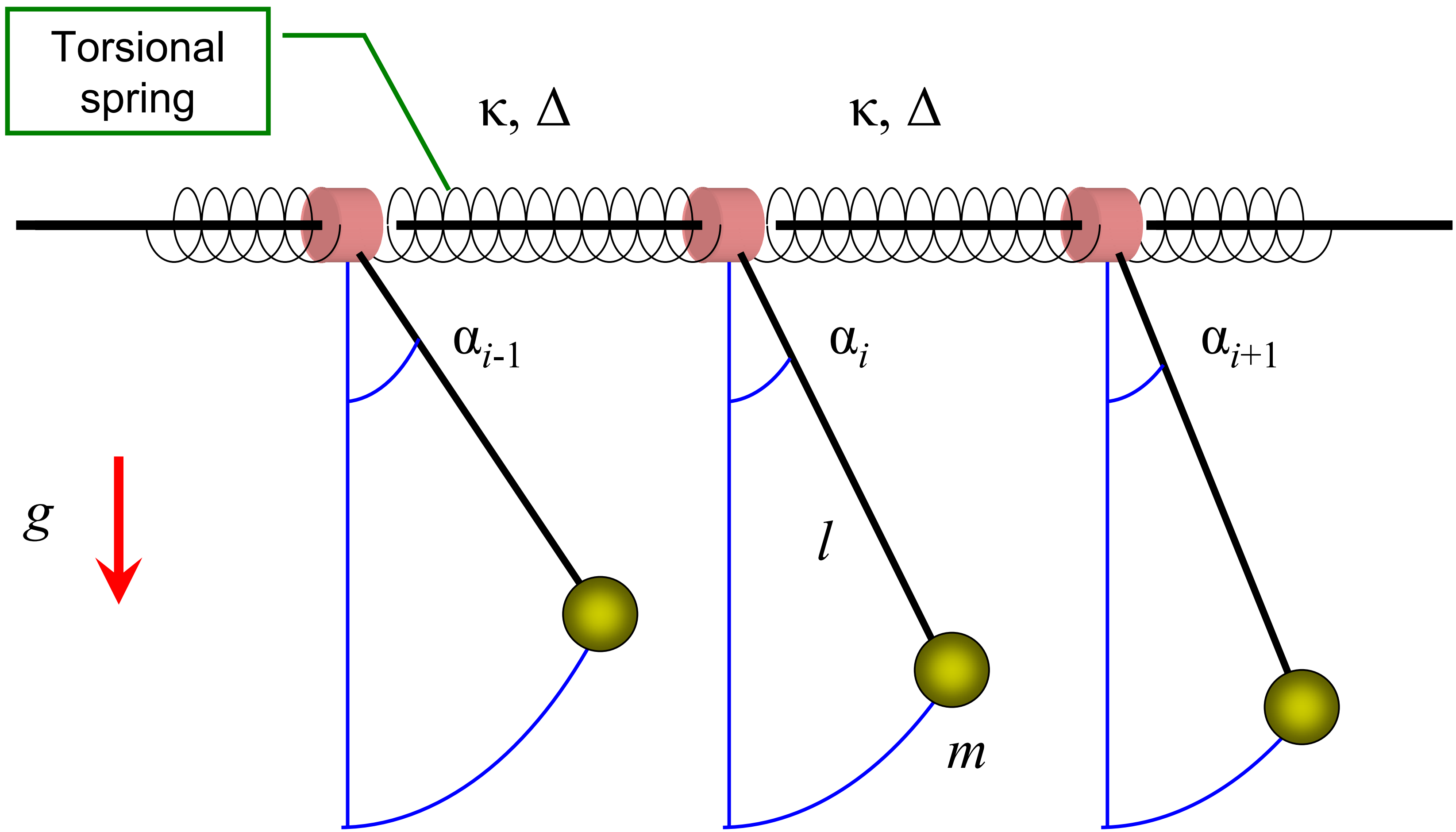
Parameters of the setup

- Number of pendula $N = 40$
- Distance between adjacent pendula $\Delta = 5$ cm
- Distance between the pivot and the bend $l = 30$ cm
- Moment of inertia of the pendulum $I = 5 \cdot 10^{-3}$ kg·m^2
- Stiffness of the rubber bend between adjacent pendula $k = 190$ N/m
- Pre-tension of the rubber bend $F_0 = 4.7$ N
- **Period of small free oscillations $T = 1.1$ s**

Solitary wave (240 fps)

Coupled pendula with axial springs

Coupled pendula with axial springs



$$M_{i,i+1} = \kappa\alpha_{i,i+1}$$

Derivation of sine-Gordon equation

$$\frac{d^2\alpha_i}{dt^2} - \frac{\kappa\Delta^2}{ml^2}\cdot\frac{\alpha_{i+1} - 2\alpha_i + \alpha_{i-1}}{\Delta^2} + \frac{g}{l}\cdot\sin\alpha_i = 0$$

$$\frac{\partial^2\alpha}{\partial t^2} - c^2\frac{\partial^2\alpha}{\partial x^2} + \Omega^2\sin\alpha = 0$$

Limiting cases of sine-Gordon equation

$$k = 0 \quad \Rightarrow \quad \frac{\partial^2 \alpha}{\partial t^2} + \Omega^2 \sin \alpha = 0$$

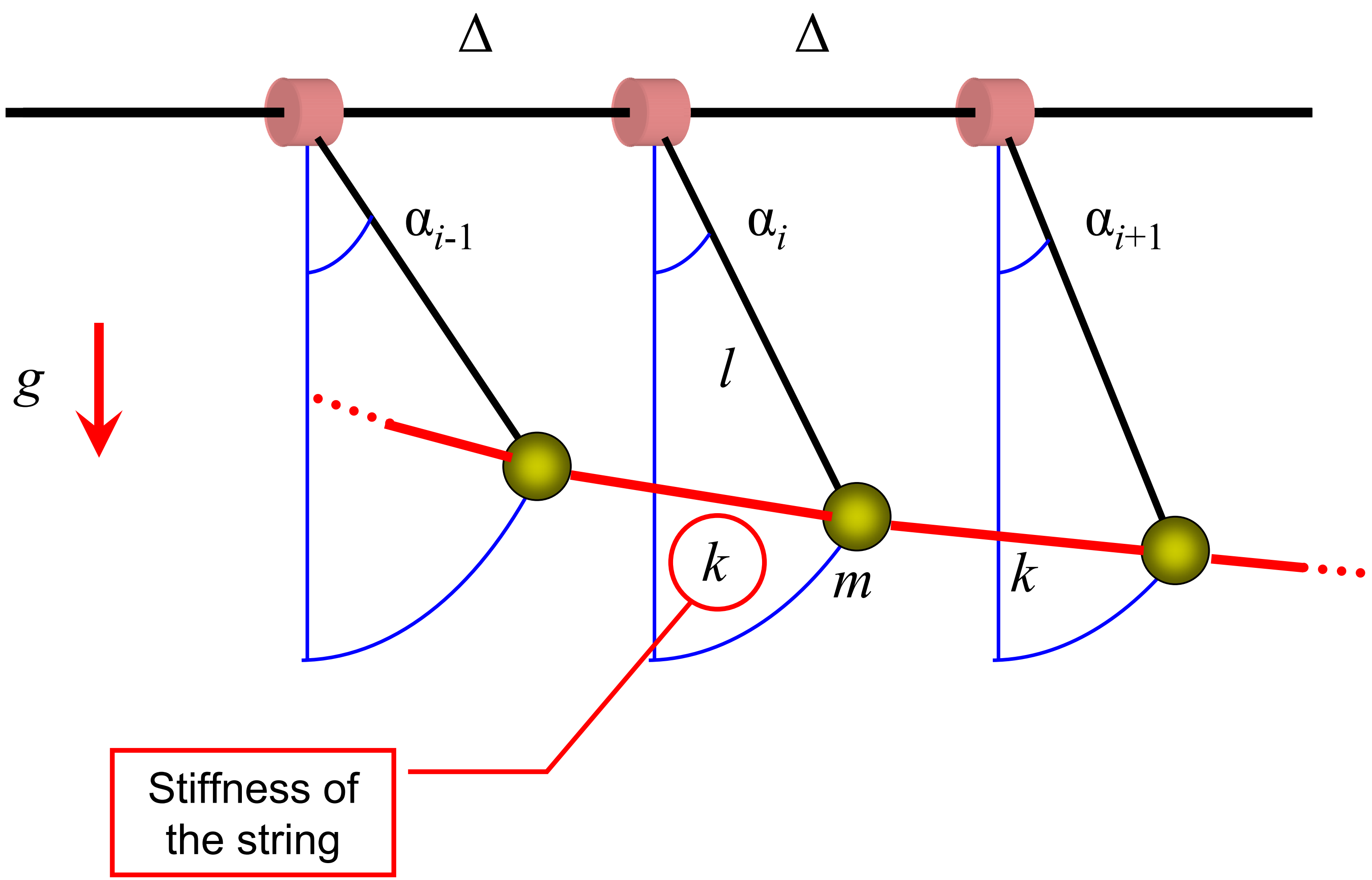
Pendulum equation

$$g = 0 \quad \Rightarrow \quad \frac{\partial^2 \alpha}{\partial t^2} - c^2 \frac{\partial^2 \alpha}{\partial x^2} = 0$$

Wave equation

Coupled pendula with the end strings

Coupled pendula with the end strings

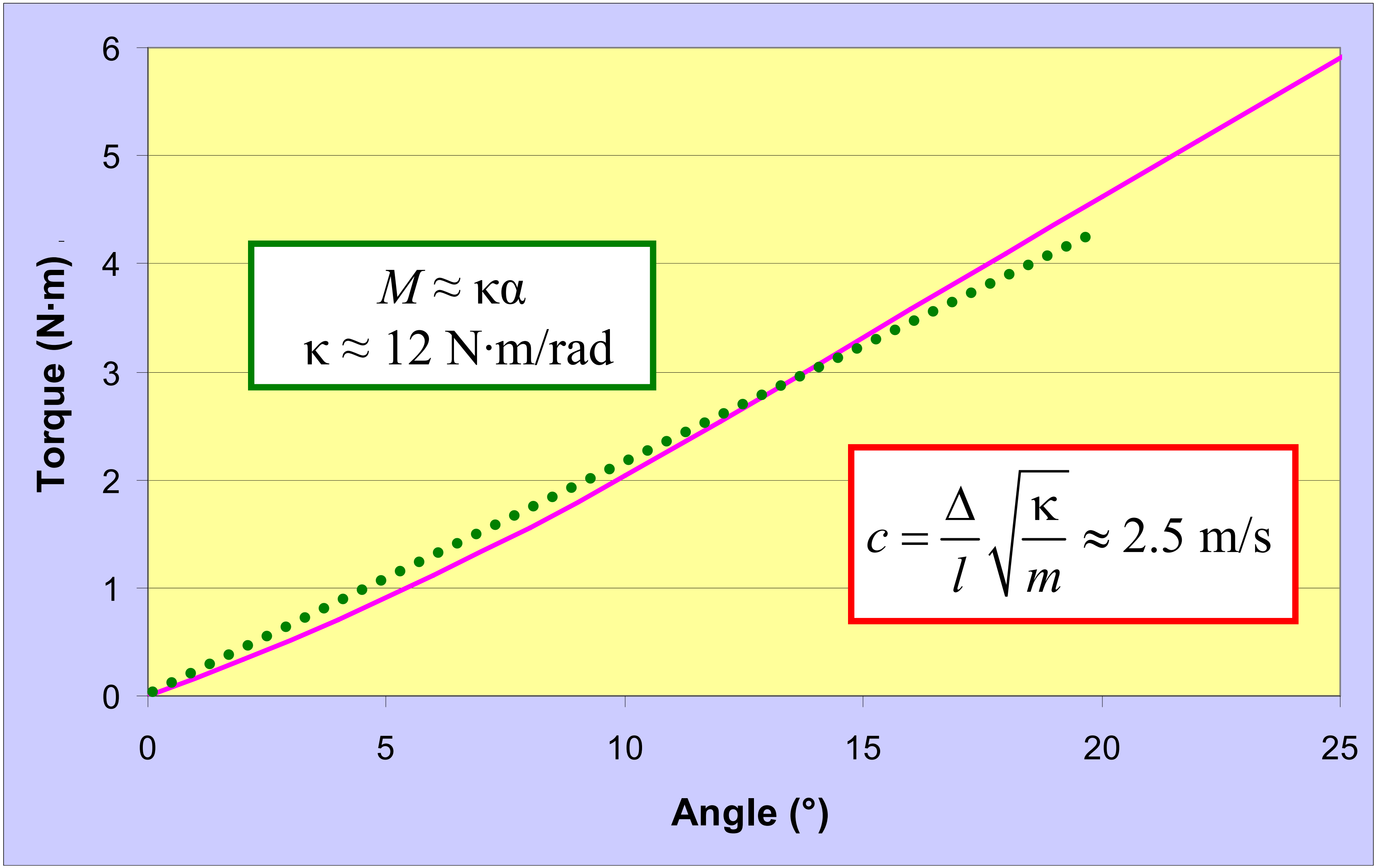


Torque between the adjacent pendula

Pre-tension

$$M_{i,i+1} = \frac{F_0 + k\left(\sqrt{\Delta^2 + \left(2l\sin\frac{\alpha_{i,i+1}}{2}\right)^2} - \Delta\right)}{\sqrt{\Delta^2 + \left(2l\sin\frac{\alpha_{i,i+1}}{2}\right)^2}} \cdot l^2 \sin\alpha_{i,i+1}$$

Torque between the adjacent pendula



Small-amplitude waves

Klein-Gordon equation

Instead of $\sin\alpha$

$$|\alpha| \ll 1 \quad \Rightarrow \quad \frac{\partial^2 \alpha}{\partial t^2} - c^2 \frac{\partial^2 \alpha}{\partial x^2} + \Omega^2 \alpha = 0$$

Wave solution

$$\alpha(x,t) = \alpha_0 \sin(kx - \omega t + \varphi)$$

$$k = \frac{2\pi}{\lambda}$$

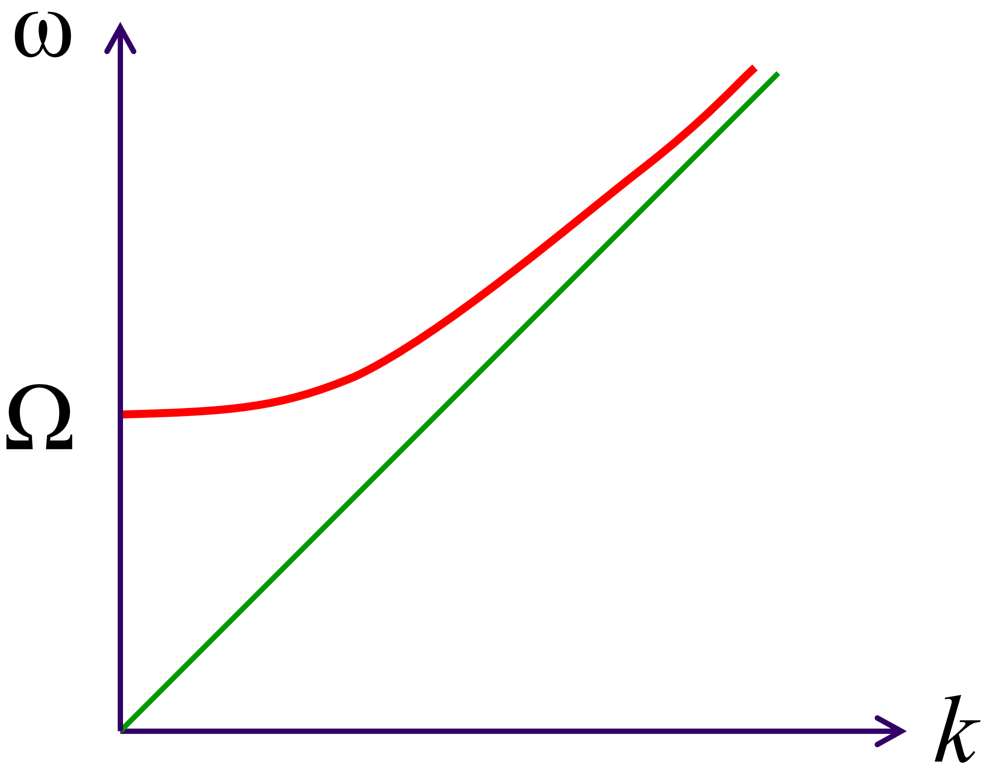
Circular wavenumber

$$\omega = 2\pi f$$

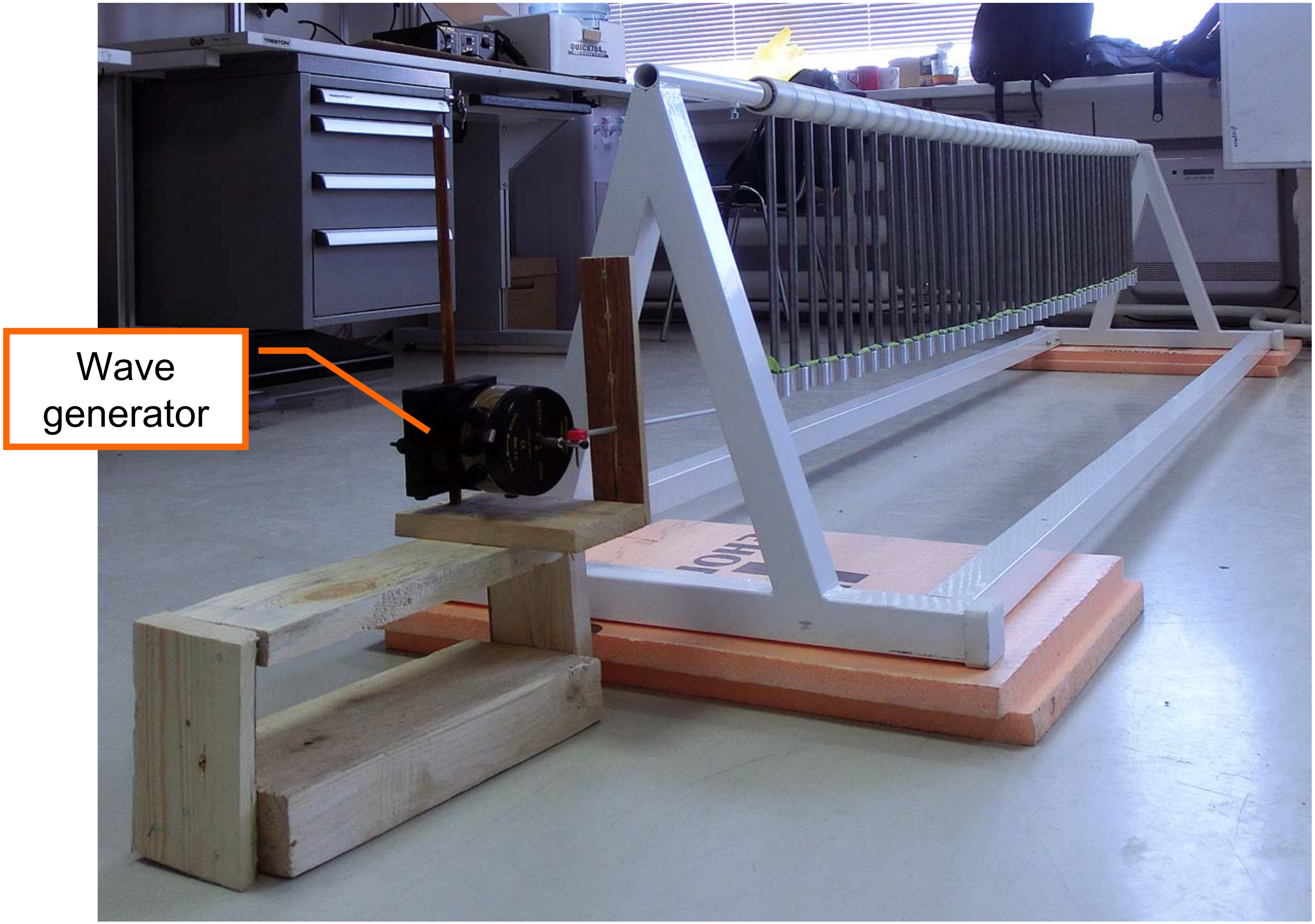
Circular frequency

Dispersion relation

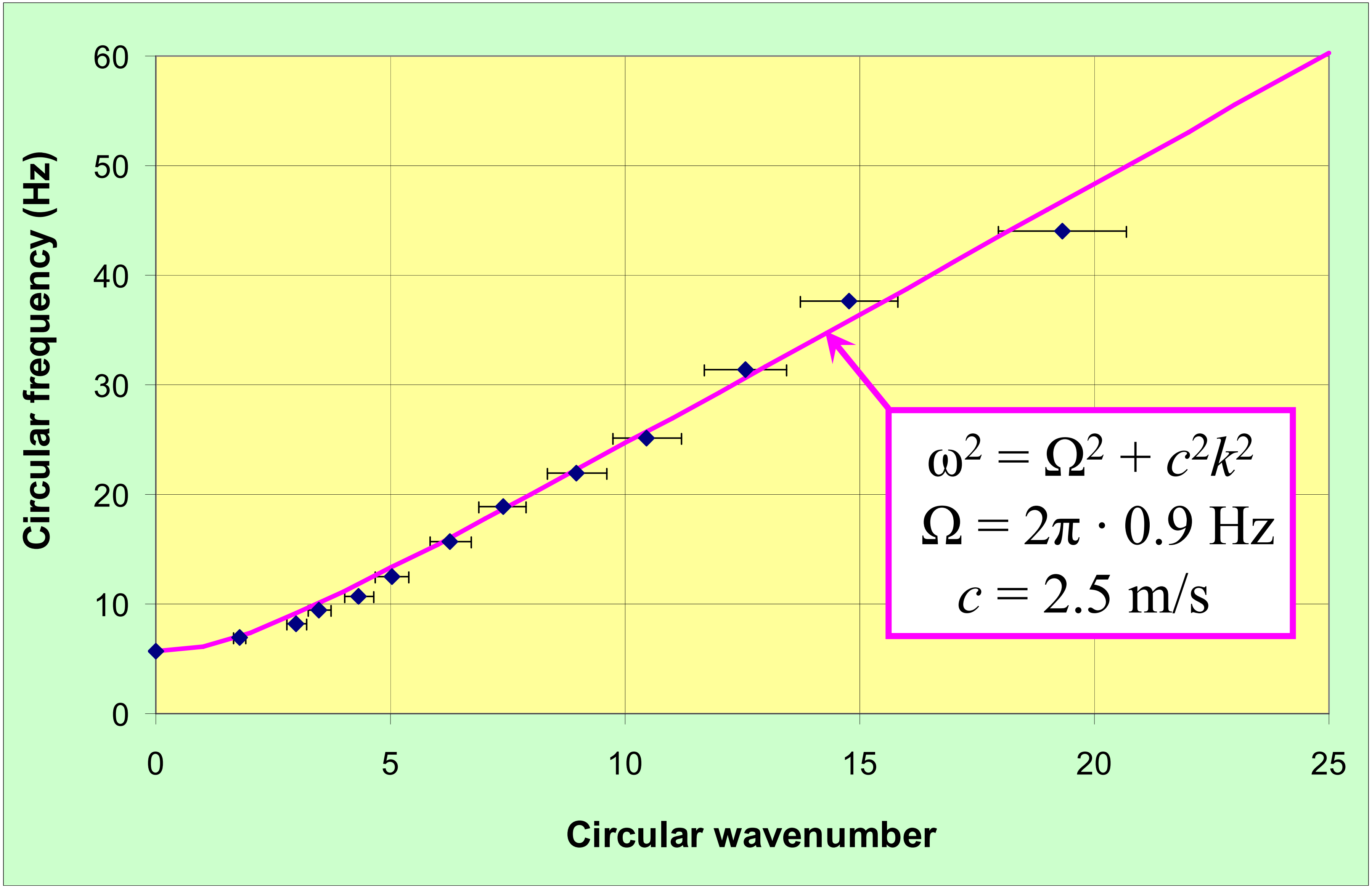
$$\omega^2 = \Omega^2 + c^2 k^2$$



Experimental setup



Dispersion relation



Sine-Gordon solitons

Definition of a soliton

SOLITON is a solitary wave which propagates in a nonlinear medium.

Soliton can travel over very large distances without changing its shape.

Conditions of a soliton solution

- A simple class of sine-Gordon solutions are **traveling waves**, which have a constant shape and propagate at a constant speed.
- **Soliton** is a special case of a traveling wave. In this case on both sides far from the disturbance the pendula hang down and do not swing.

Soliton solution

$$\alpha(x,t) = 4\,\mathrm{arctg}\left(\exp\left(\pm\frac{x \pm vt}{L}\right)\right)$$

$$L = L_0\sqrt{1-(v/c)^2}$$

$$L_0 = c/\Omega$$

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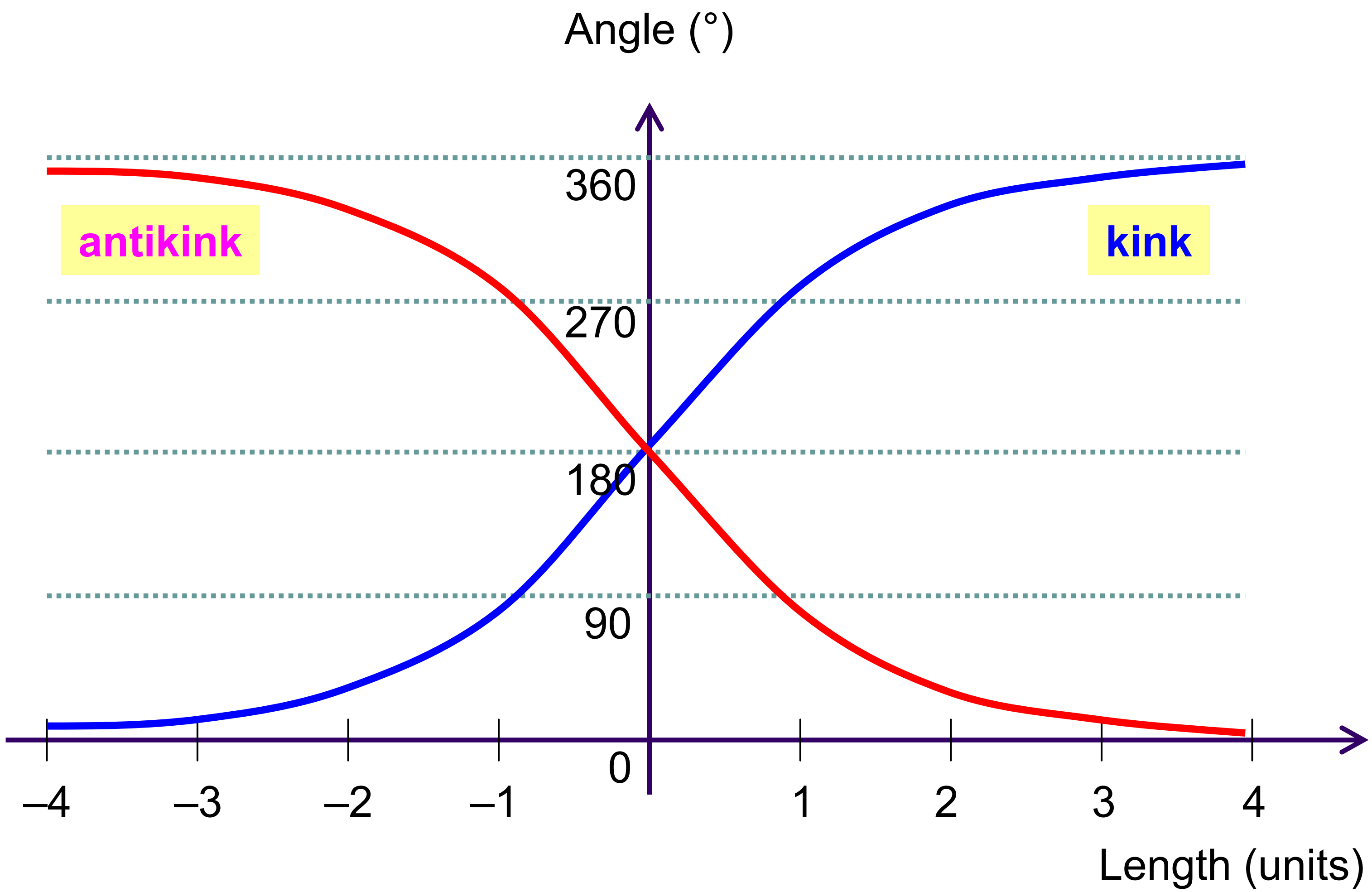
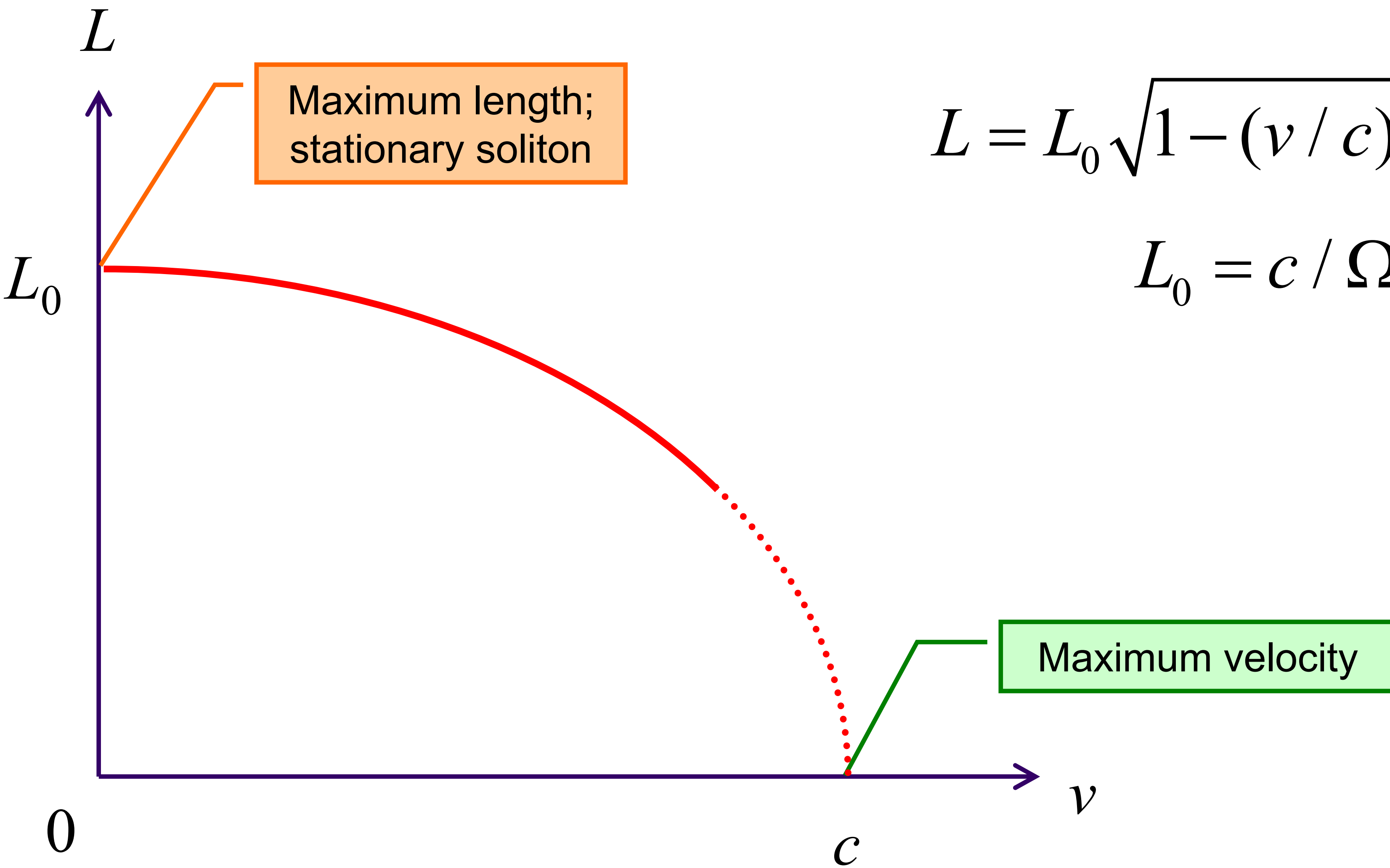
Soliton solution



Length vs. velocity

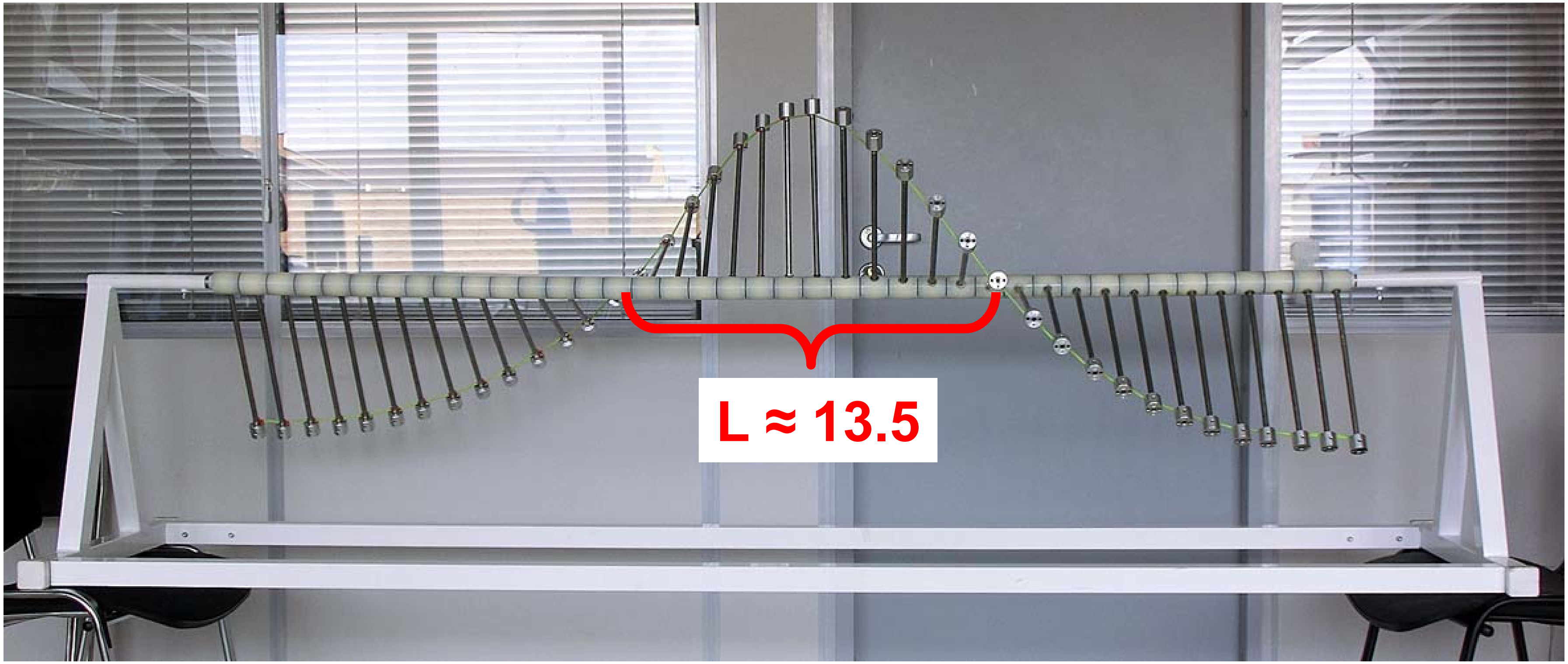


$$L = L_0 \sqrt{1 - (v/c)^2}$$

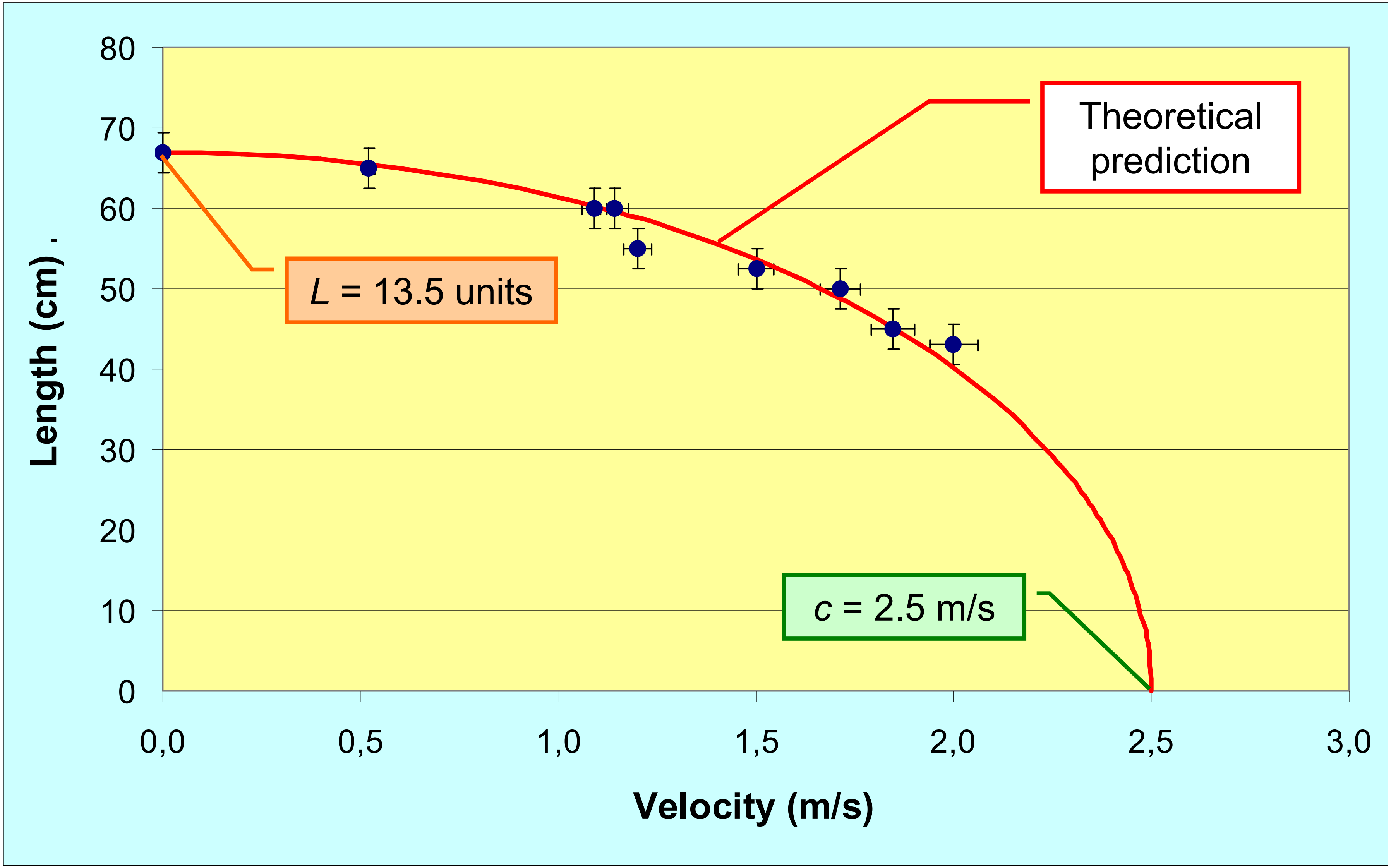
$$L_0 = c / \Omega$$

Experiment

Stationary soliton

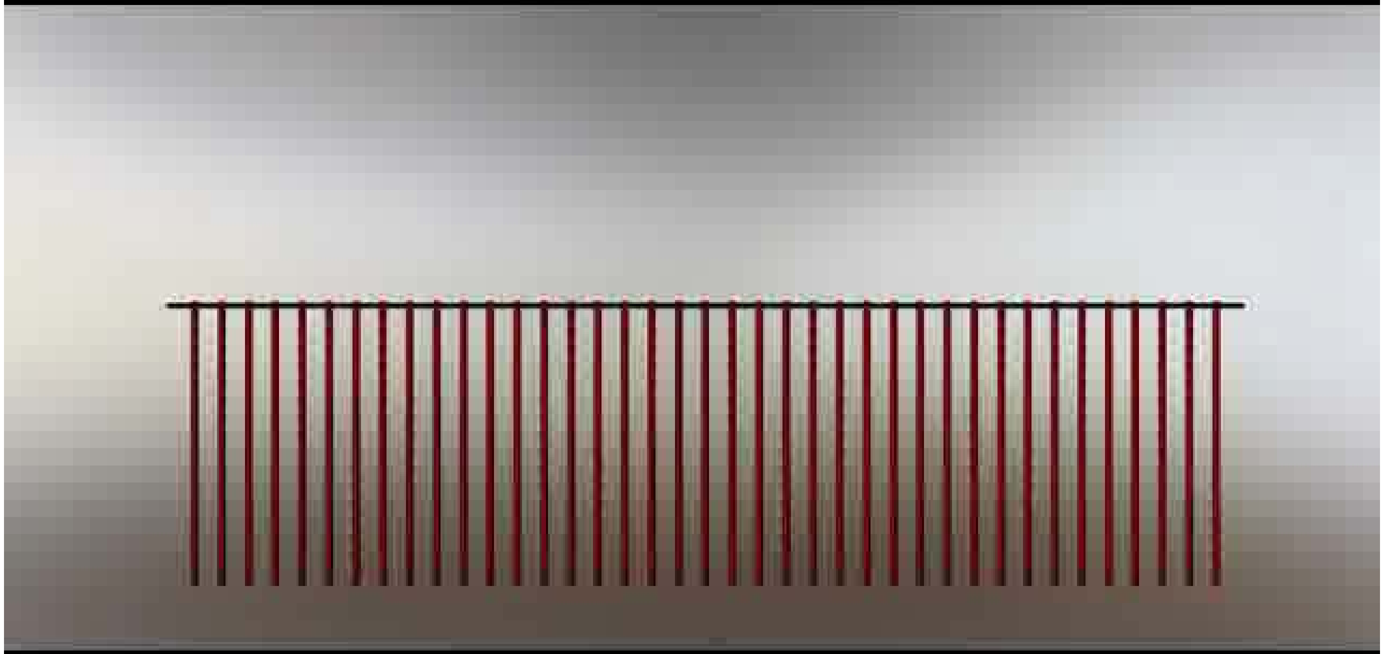


Length vs. velocity



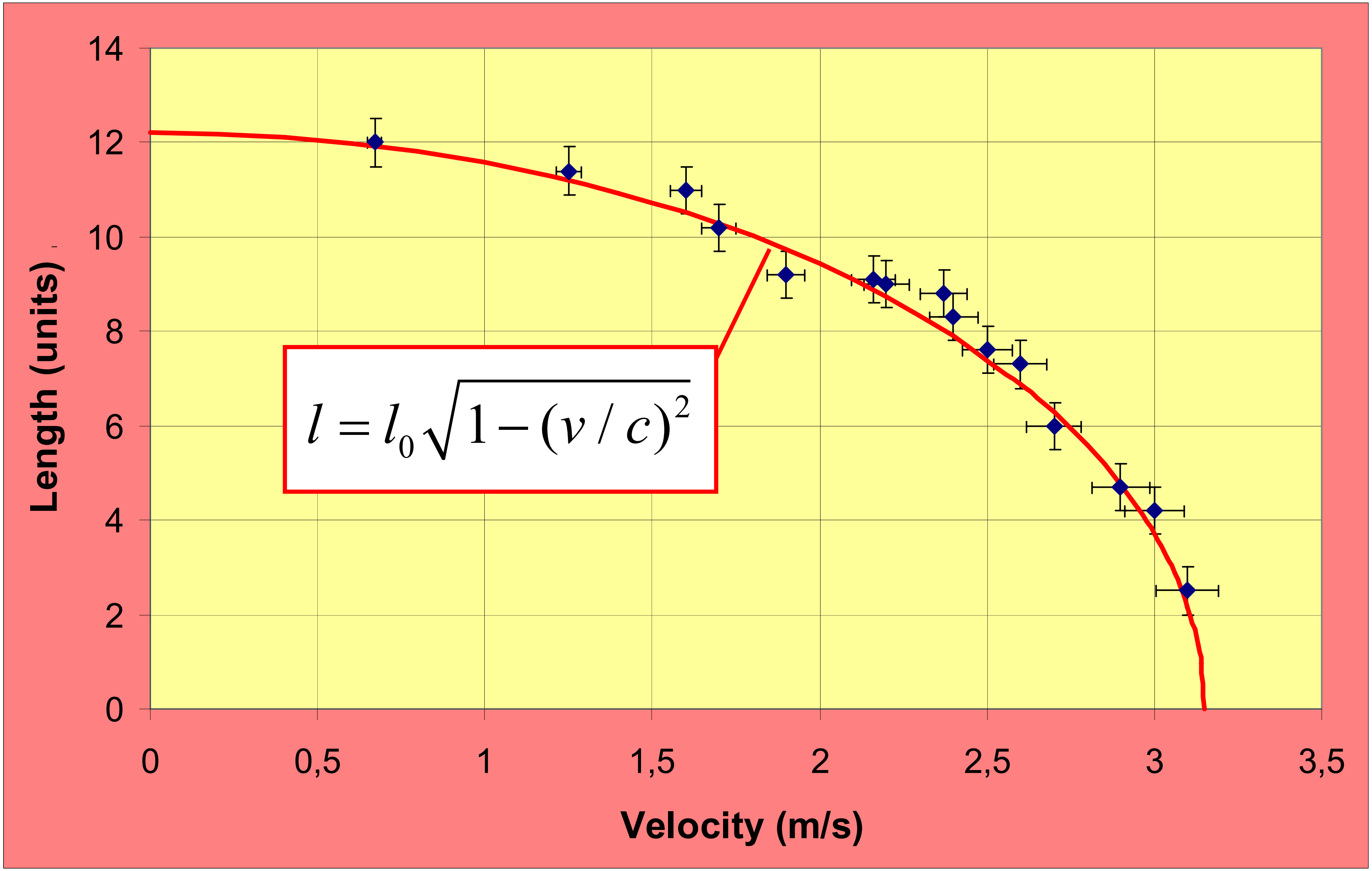
Computer simulation

“Solid Works” simulation



System without pre-tension

Length vs. velocity



Summary

Conclusions

- The system of pendula with end strings generally behaves the same way as the system of pendula with axial torsional springs. Therefore we can use sine-Gordon equation for its description.
- The length of the sine-Gordon soliton depends on its velocity. The stationary soliton has the largest length. The faster moving soliton, the shorter its length. Limiting velocity of the soliton is equal to the speed of short waves of small amplitude.

References

- Filippov A. T. (2000) *The versatile soliton*.
- Scott A. C. (1969) “A non-linear Klein-Gordon equation”*. Am. J. Phys*., **37**, 52–61.

Thank you for your attention!