SLOVAKIA
IYPT '13

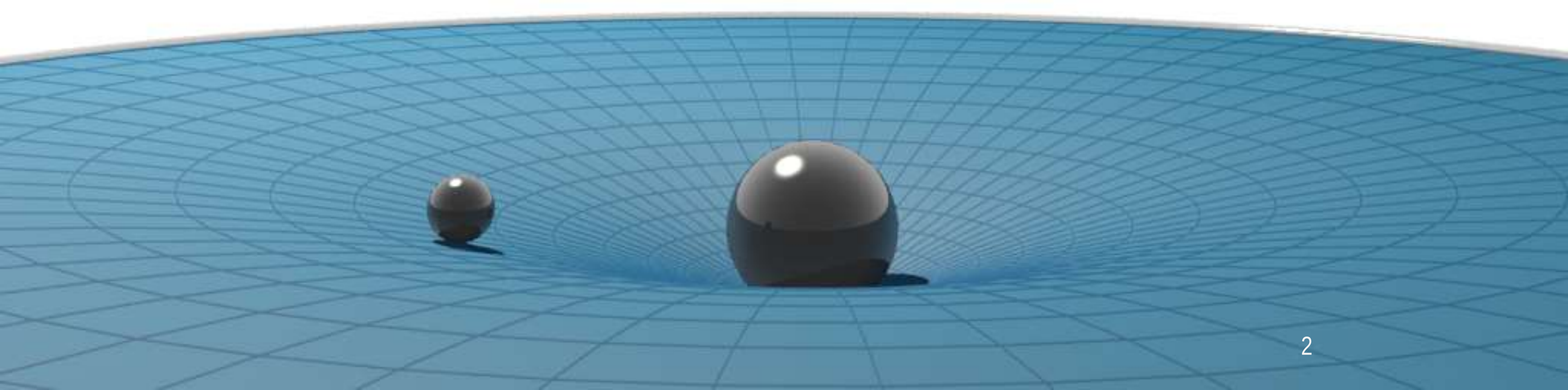
2

Elastic Space

Kamila Součková

2 Elastic Space

The dynamics and apparent interactions of massive balls rolling on a stretched horizontal membrane are often used to illustrate gravitation. Investigate the system further. Is it possible to define and measure the apparent "gravitational constant" in such a "world"?



…"illustrate gravitation"…

when viewed from the top, the objects should move as if interacting gravitationally

The Correct Shape

both systems:

$$E_k \leftrightarrow E_p$$

rubber sheet

$$\frac{1}{2}mv^2 \leftrightarrow mgh(r)$$

gravity

$$\frac{1}{2}mv^2 \leftrightarrow -\kappa\frac{mM}{r}$$

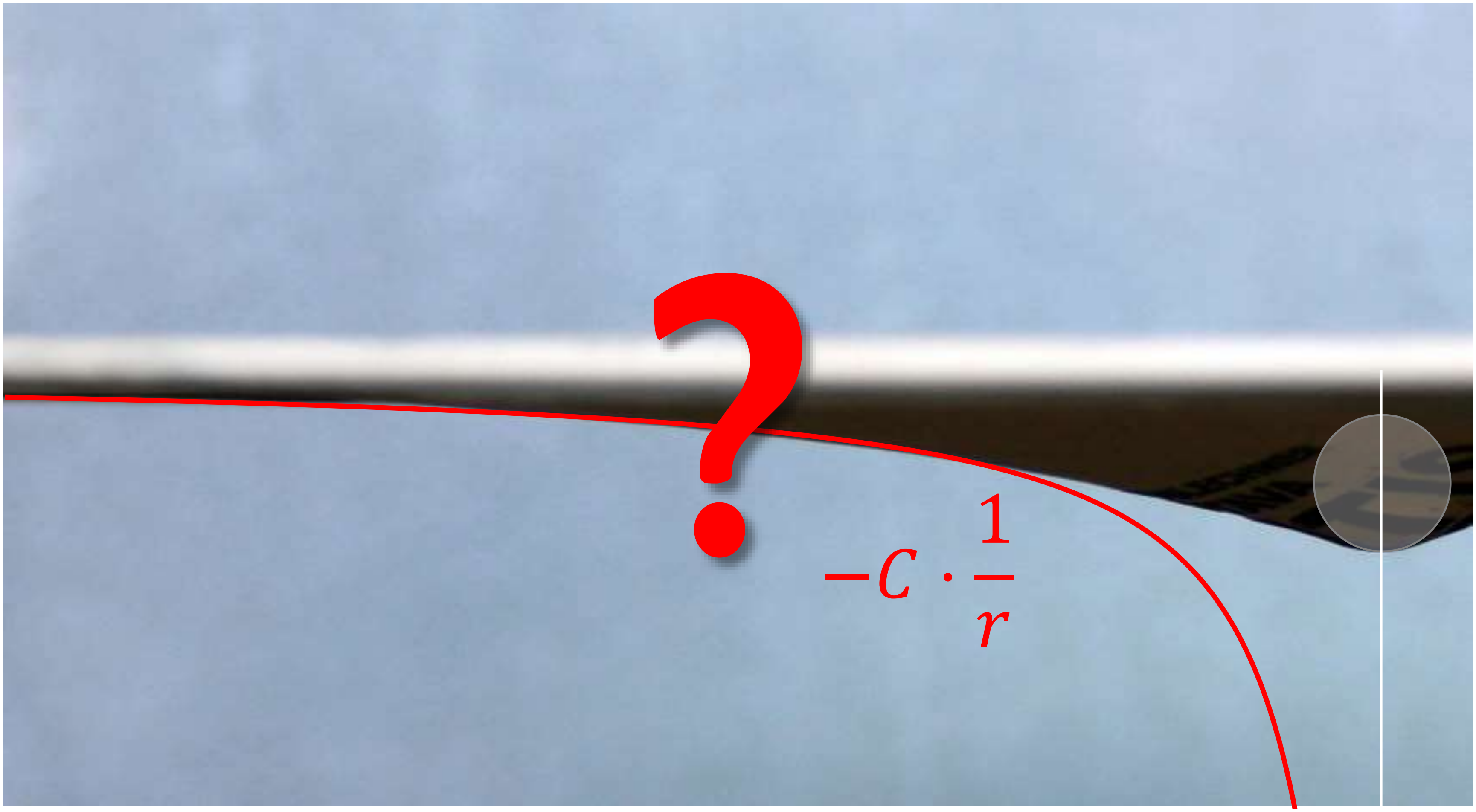
$$F_g = -\kappa\frac{Mm}{r^2}$$

$$\boxed{h(r) = -\kappa_m M\frac{1}{r}} = \phi = \frac{E_p}{m}$$

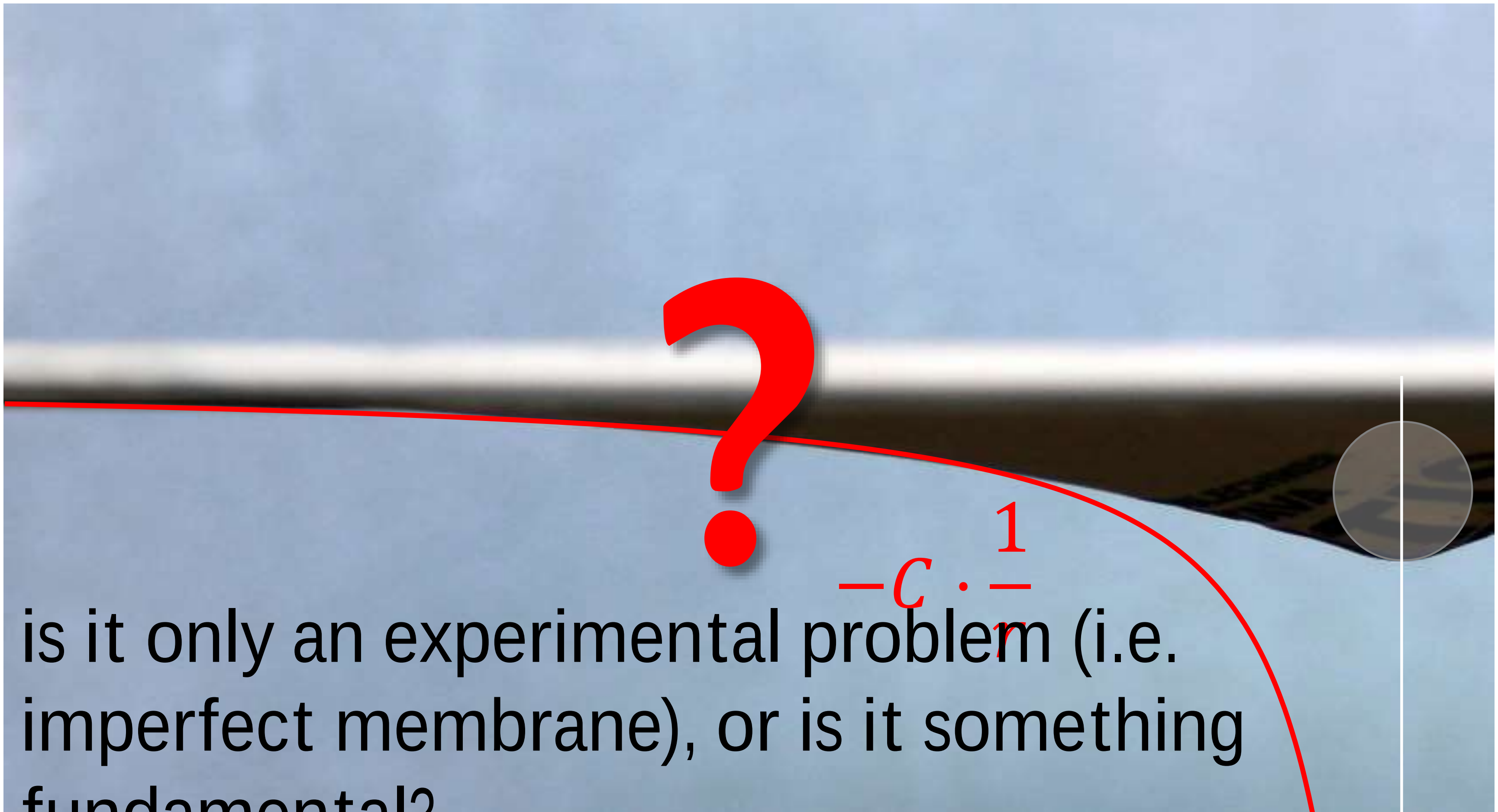
grav. potential

$h(r)$: membrane height at radius r

Is the Shape Correct?



Is the Shape Correct?



is it only an experimental problem (i.e. imperfect membrane), or is it something fundamental?

PHYSICS OF A RUBBER SHEET

What Does Rubber Do?

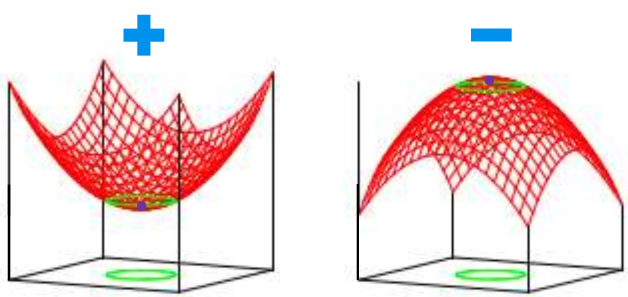
Description of Rubber Sheet Curvature

- based on *Feynman, R. P.: The Feynman Lectures On Physics, Vol. 2, Ch. 12*:

$$\nabla^2 h = \frac{g}{k}\rho$$

Laplace operator

valid for small deformations only;
derivation in Appendix



h: membrane height at radius $\boldsymbol{r}$
ρ: distribution of mass
k: membrane stiffness
g: gravitational acceleration

Rubber Sheet vs. Gravity

rubber sheet shape

gravitational potential

$$\nabla^2 h = \frac{g}{k}\rho$$

Rubber Sheet vs. Gravity

rubber sheet shape = gravitational potential

$$\nabla^2 h = \boxed{\frac{g}{k}} \rho \qquad \nabla^2 \phi = \boxed{4\pi\kappa} \rho$$

(derivations in Appendix)

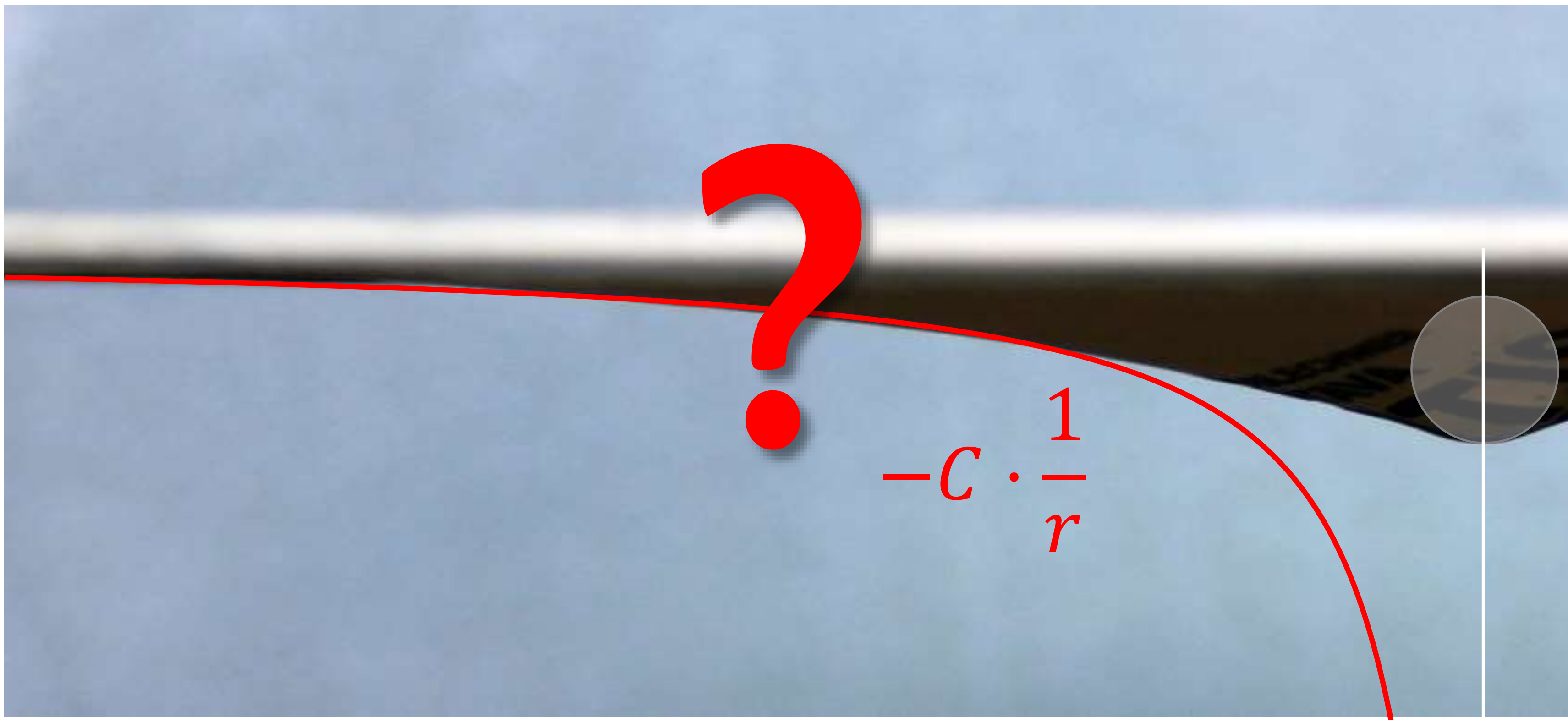
Laplace operator

h: membrane height
ρ: distribution of mass
k: membrane stiffness
g: gravitational acceleration

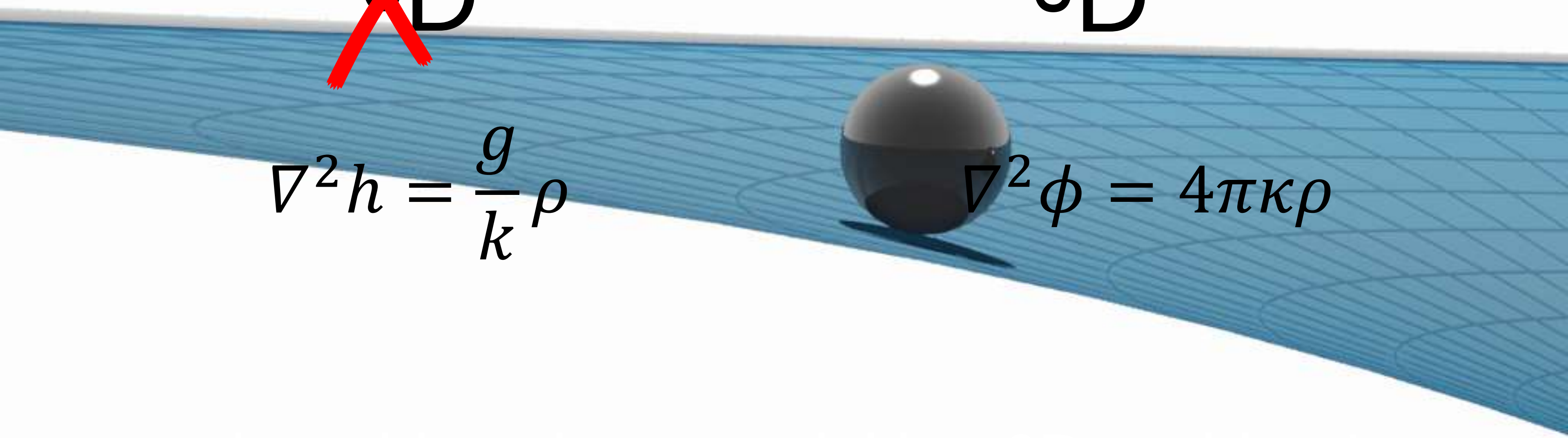
ρ: distribution of mass
ϕ: gravitational potential
κ: gravitational constant

Rubber Sheet vs. Gravity

rubber sheet shape = gravitational potential



rubber sheet shape $\neq$ gravitational potential



$\cancel{{}^{3}}$D

$$\nabla^2 h = \frac{g}{k}\rho$$

^{3}D

$$\nabla^2 \phi = 4\pi\kappa\rho$$

the rubber sheet model is a 2D problem!

GRAVITY IN 2 DIMENSIONS

Rubber Sheet and Gravity Equivalence

The Form of Newton's Law of Gravitation

- why $F_g \propto \frac{1}{r^2} \Rightarrow \phi \propto -\frac{1}{r}$?
  - depends on the dimensionality of the world

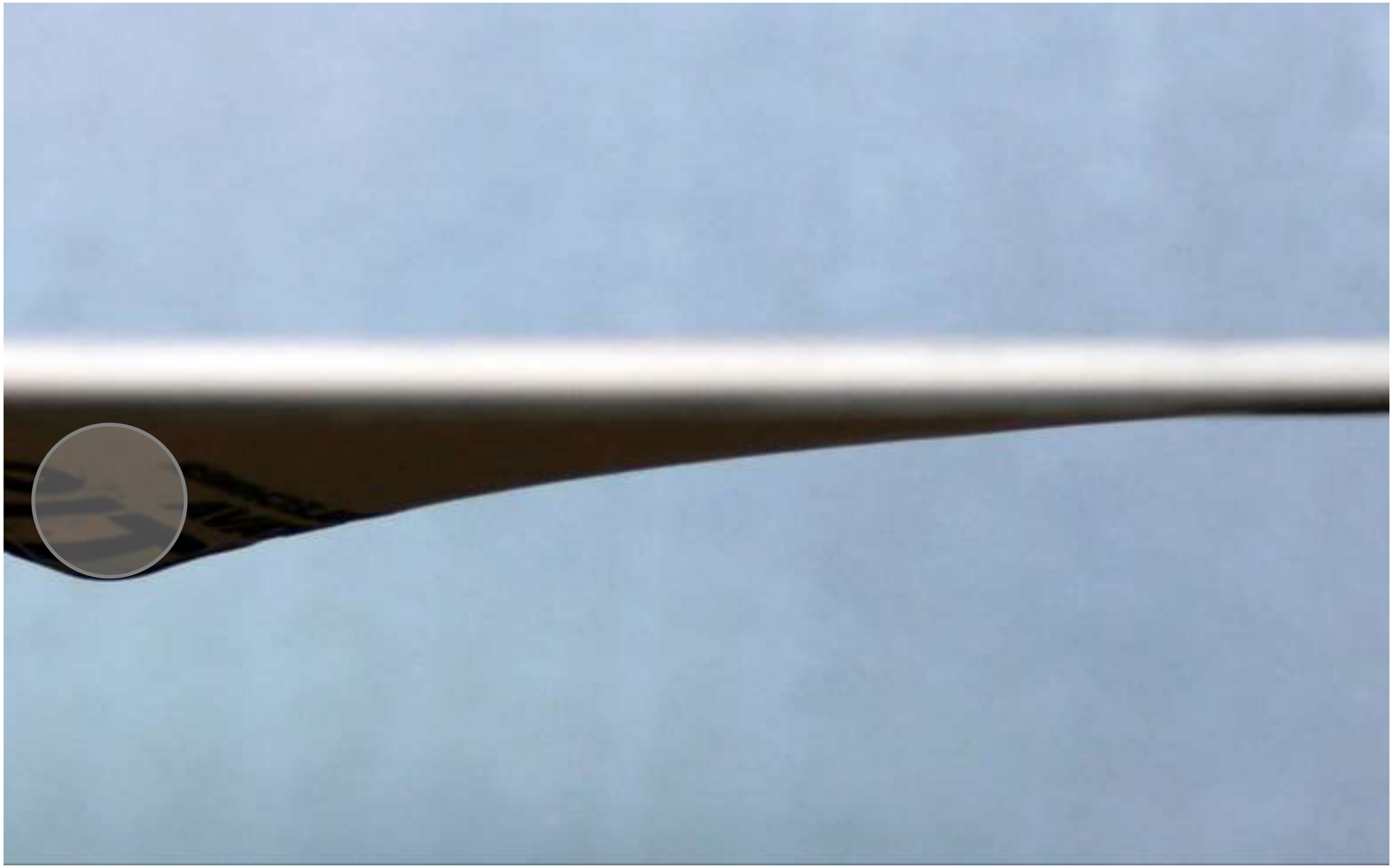
informal reasoning: $F_g \propto$ intensity

- intensity in 3D $\propto \frac{1}{r^2}$
- intensity in 2D $\propto \frac{\mathbf{1}}{\boldsymbol{r}}$ $\Rightarrow$ $\boxed{F_g \propto \frac{1}{r}}$

(formal derivation based on Divergence Theorem in Appendix)

2D $F_g \propto \frac{1}{r}$

$$2\text{D} : F_g \propto \frac{1}{r} \Rightarrow \boxed{\phi \propto \log\frac{r}{R_0}}$$



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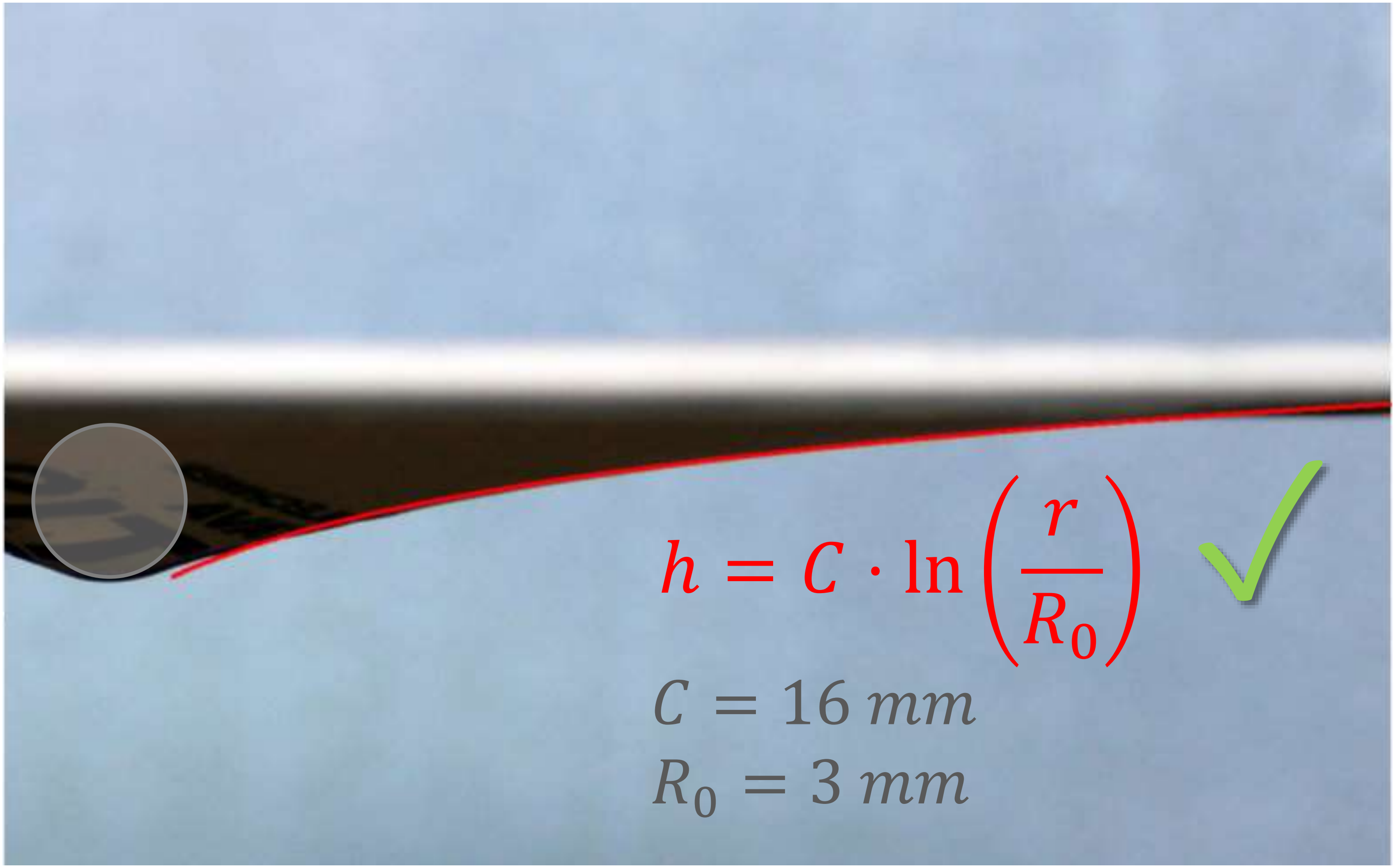

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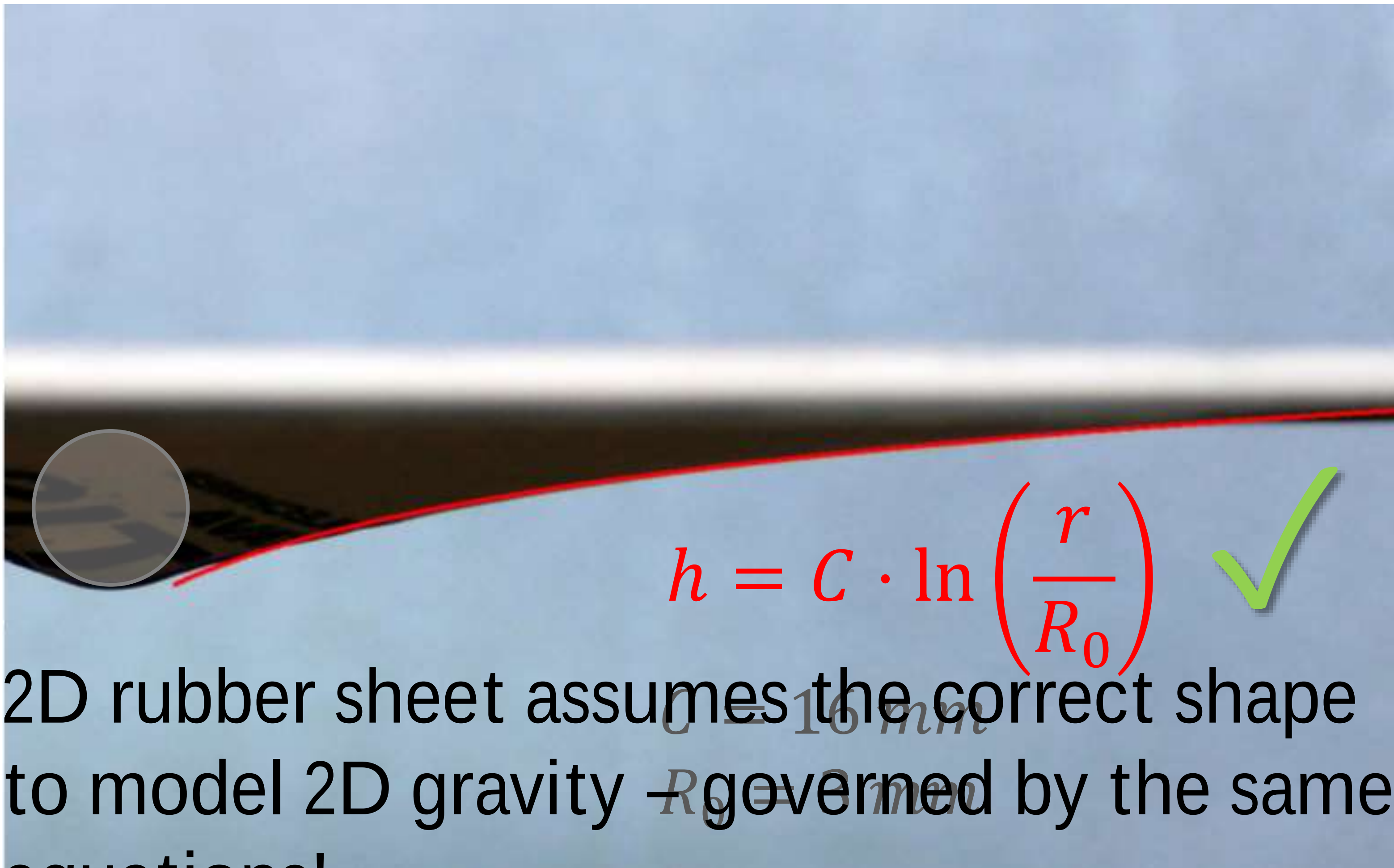


2D rubber sheet assumes the correct shape to model 2D gravity – governed by the same equations!

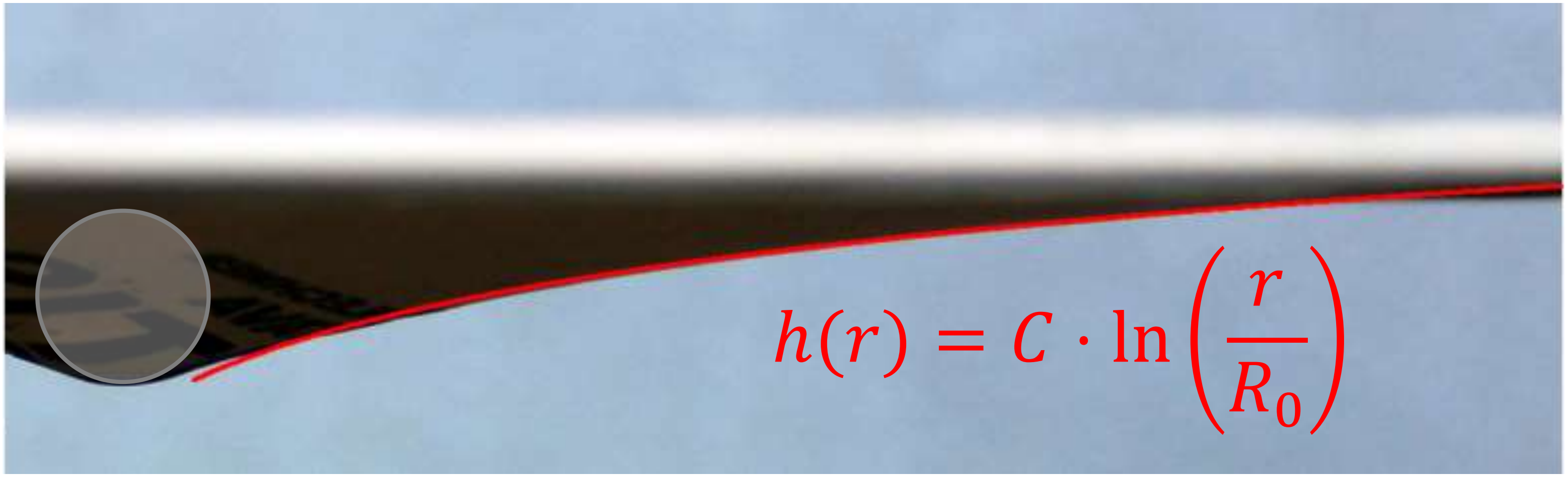
FINDING THE GRAVITATIONAL CONSTANT

Where to get κ_m?

- from the membrane shape (potential):

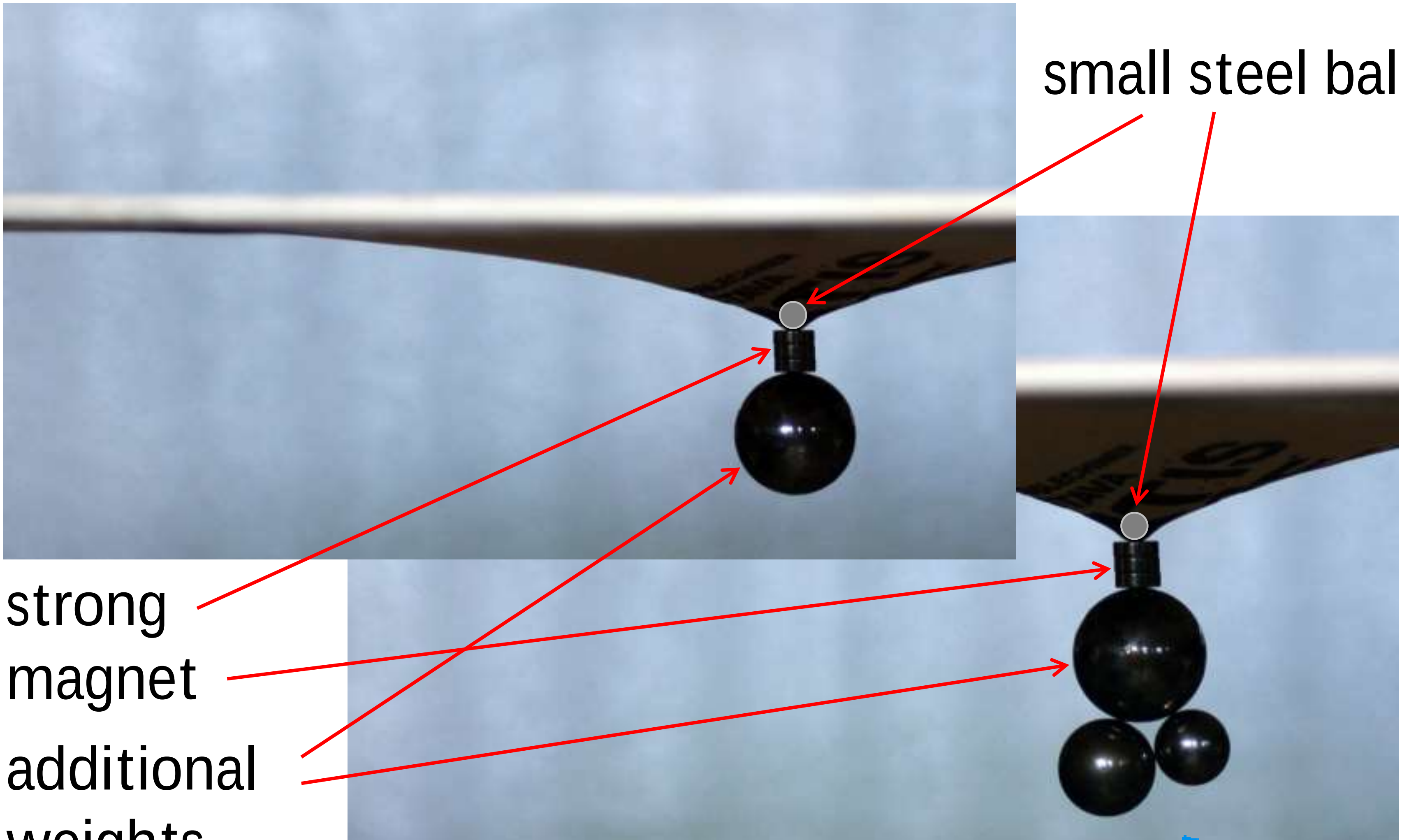
$$\phi(r) = 2\kappa_m M \ln\left(\frac{r}{R_0}\right)$$

(by solving $\nabla^2 h = \frac{g}{k}\rho$ using a Green's function)



$$\Rightarrow \quad \kappa_m = \frac{C}{2M}$$

Experiment: changing mass, keeping diameter



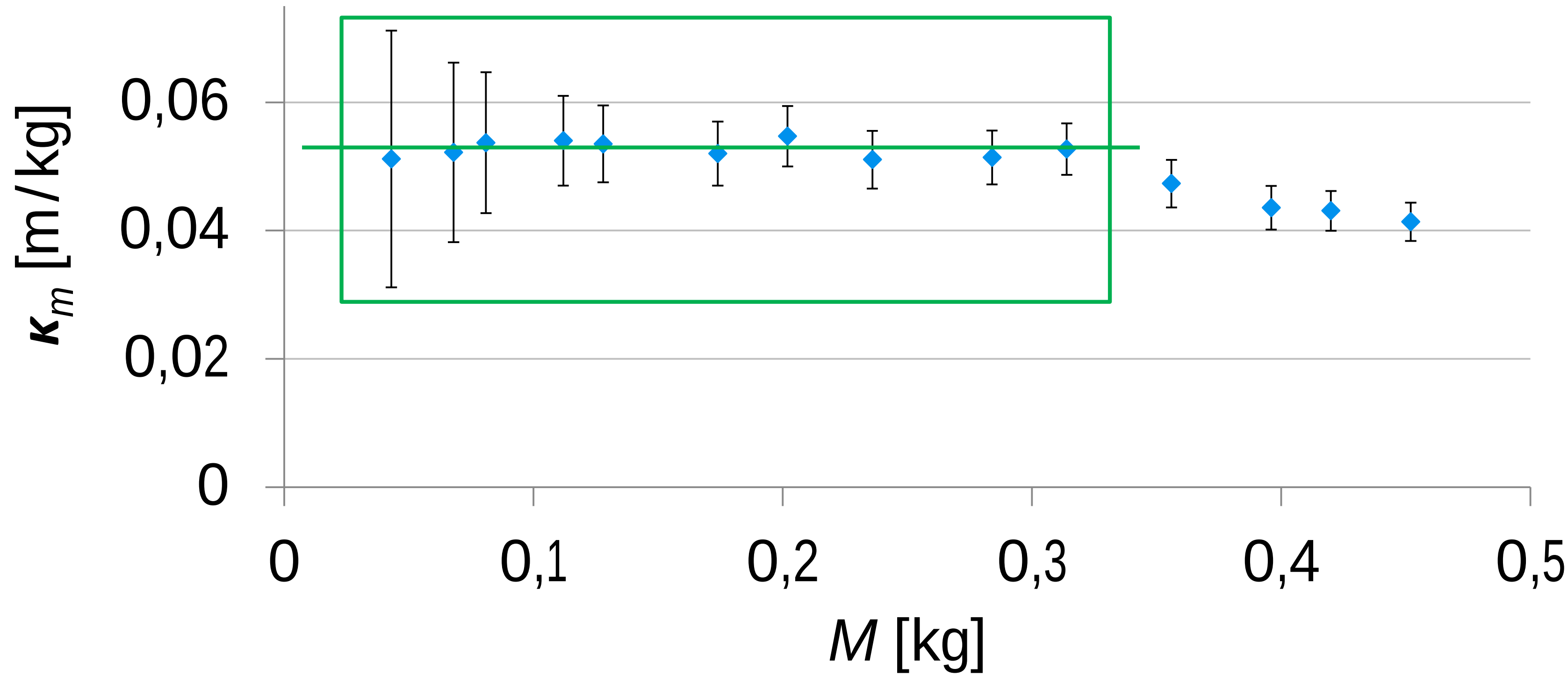
small steel ball

strong magnet

additional weights

Results

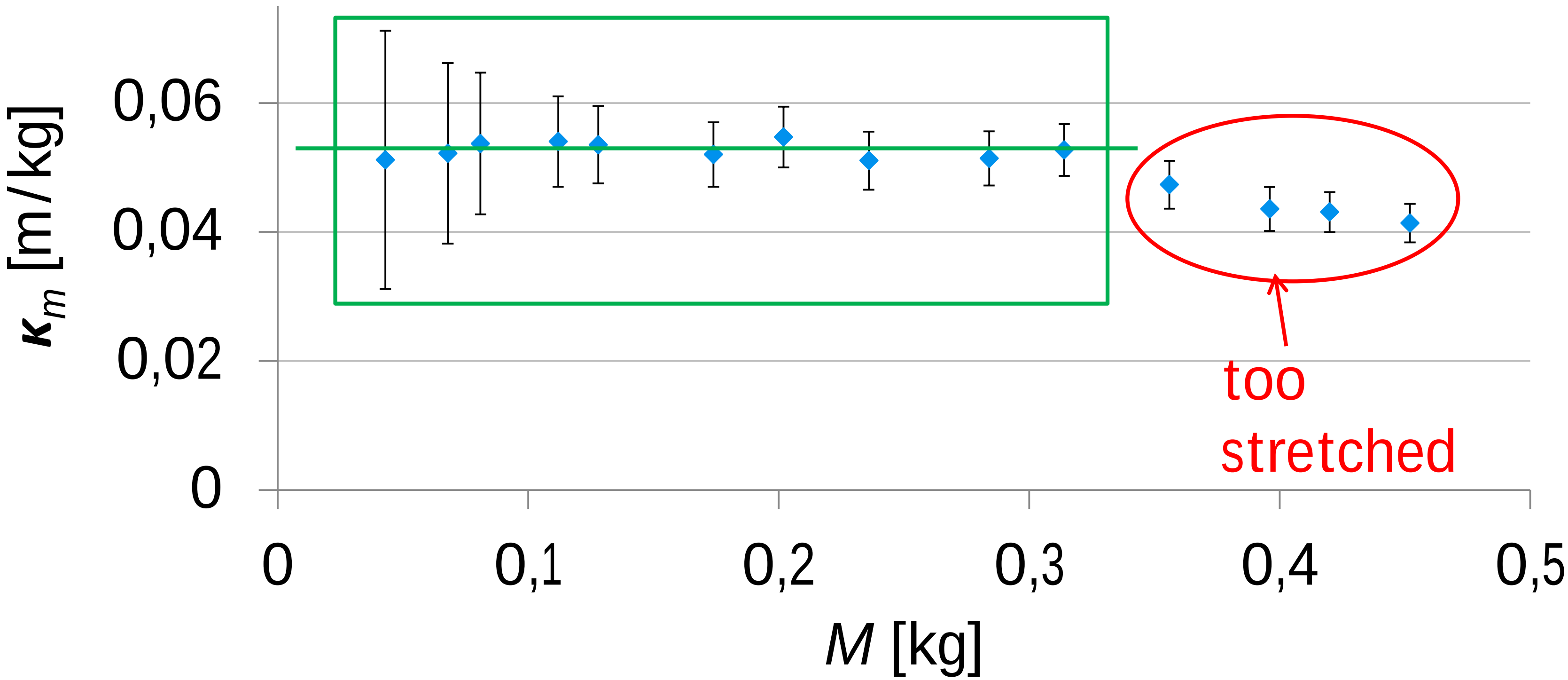
$$u(r) = C \cdot \ln\left(\frac{r}{R_0}\right) \Rightarrow \kappa_m = \frac{C}{2M}$$



$$\kappa_m = (5.3 \pm 0.2) \cdot 10^{-2}\ \mathrm{m\,kg^{-1}}$$

Results

$$u(r) = C \cdot \ln\left(\frac{r}{R_0}\right) \Rightarrow \kappa_m = \frac{C}{2M}$$



$$\kappa_m = (5.3 \pm 0.2) \cdot 10^{-2}\ \mathrm{m\,kg^{-1}}$$

DYNAMICS

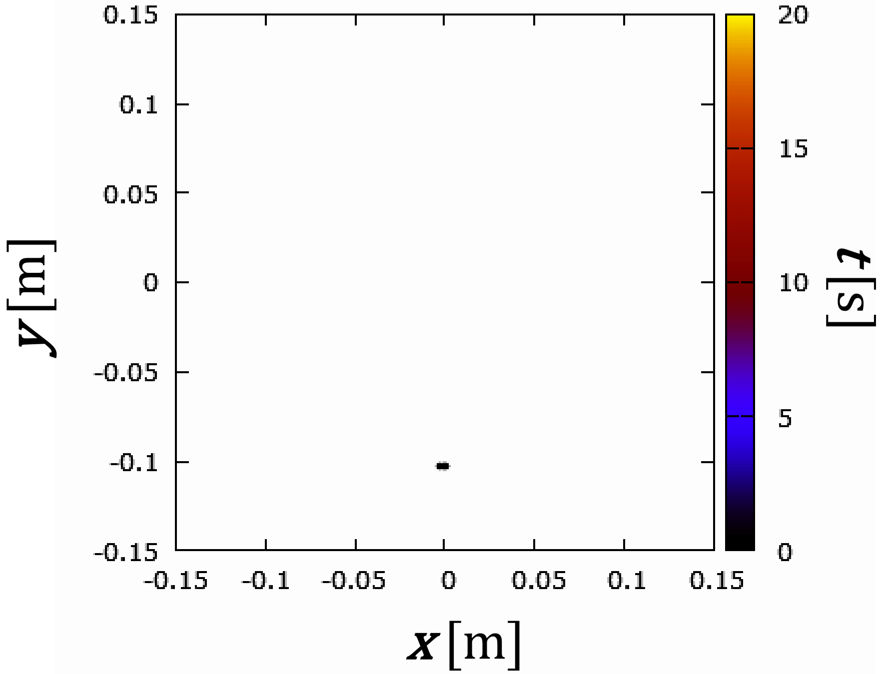
Dynamics

- the shape is correct ⇒ approx. works
- but: energy losses (friction / rolling resistance)

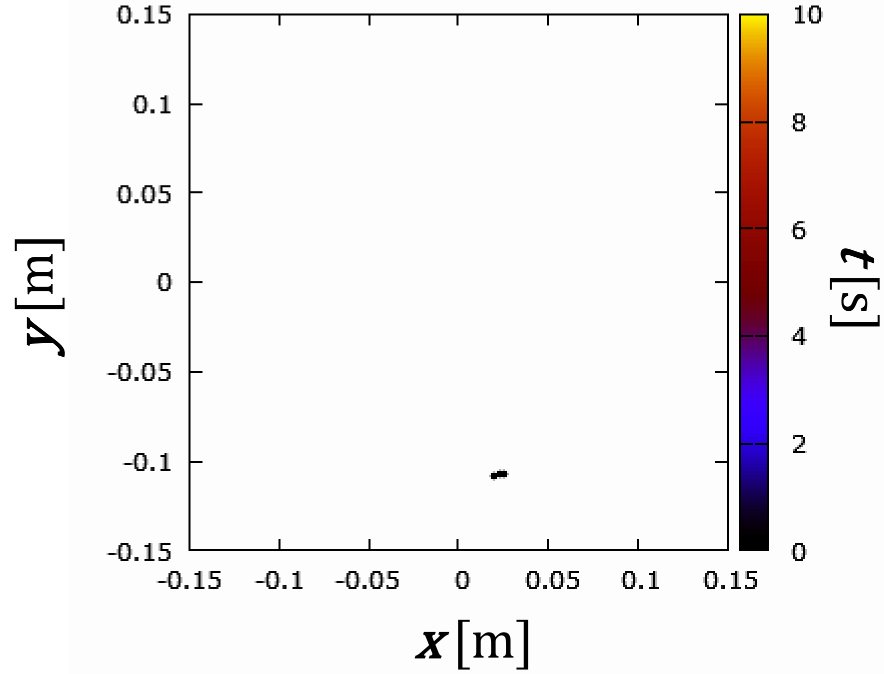
⇒ ~~conservation of mechanical energy~~

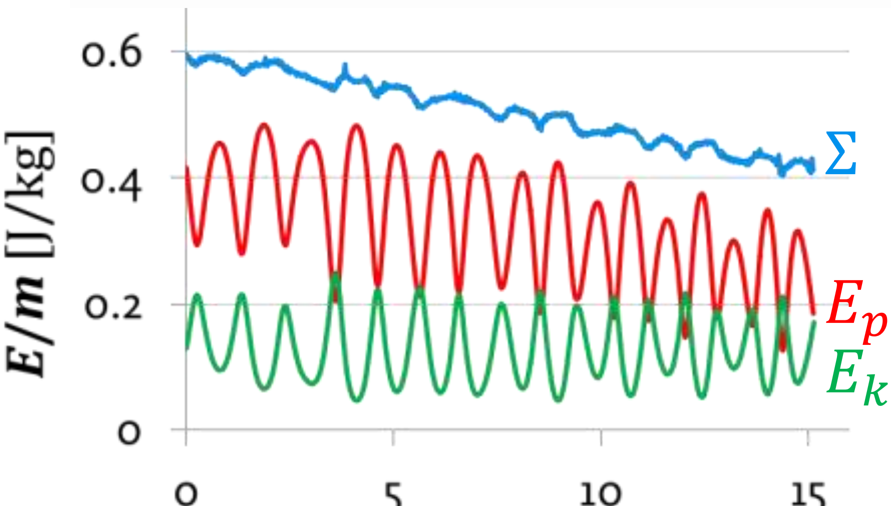
\+ elasticity: finite speed of "gravitational interaction"

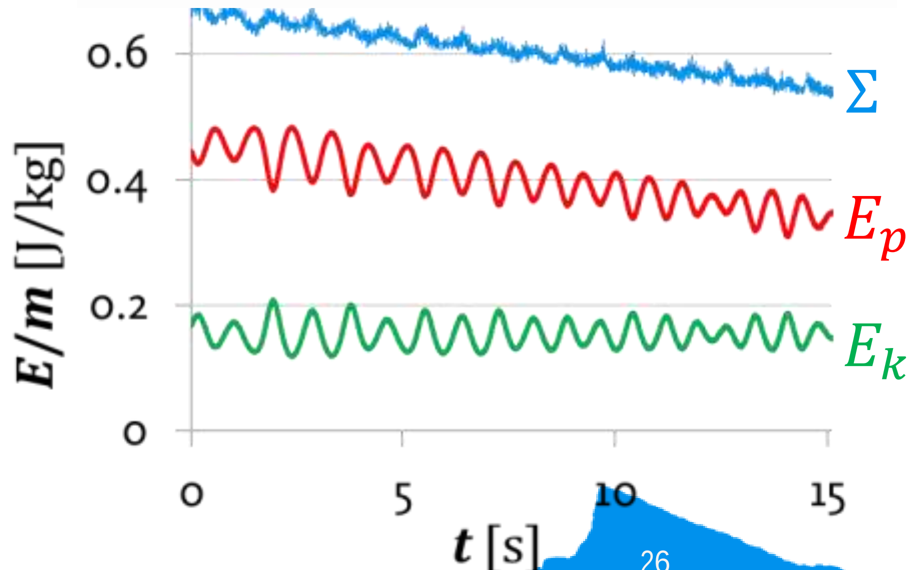
$m = 1$ g, $d = 12$ mm



$m = 2.2$ g, $d = 16$ mm



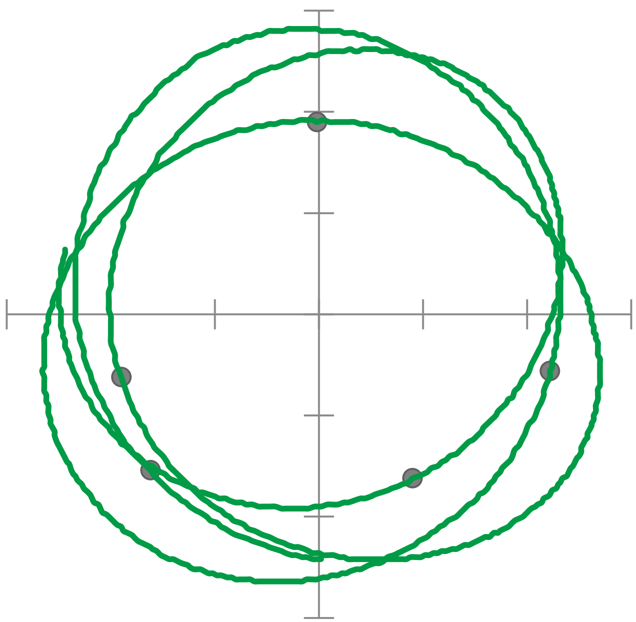




$M = 1.8$ kg

Ellipses?

- *Bertrand's theorem*: stable, closed orbits can only exist if $\phi \propto -\frac{1}{r}$ or $\phi \propto r^2 \Rightarrow$ no closed orbits here



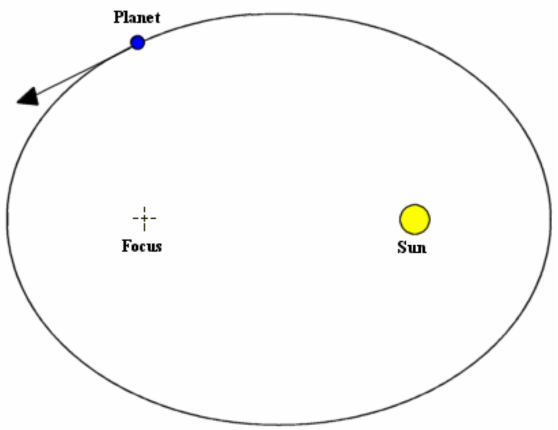
pericenter
trajectory

$M = 1.8$ kg
$m = 9.7$ g
$r_{init} = 13.4$ mm

Johnson, Porter Wear (2010). Classical Mechanics With Applications

Kepler's Laws

1st Law



X

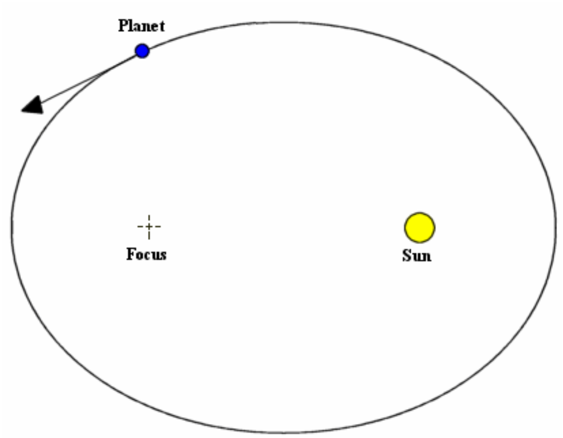
no elliptical orbits

2nd Law

3rd Law

Kepler's Laws

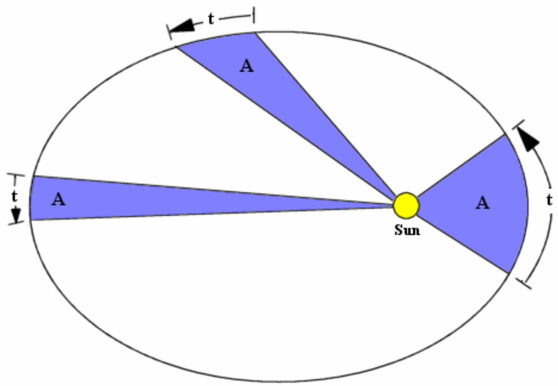
1st Law



✗

no elliptical orbits

2nd Law



theoretically: ✓

(conservation of momentum)

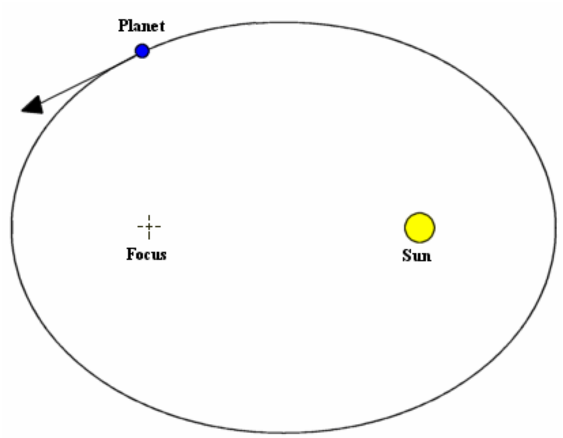
experiment: ✗

(energy losses – friction)

3rd Law

Kepler's Laws

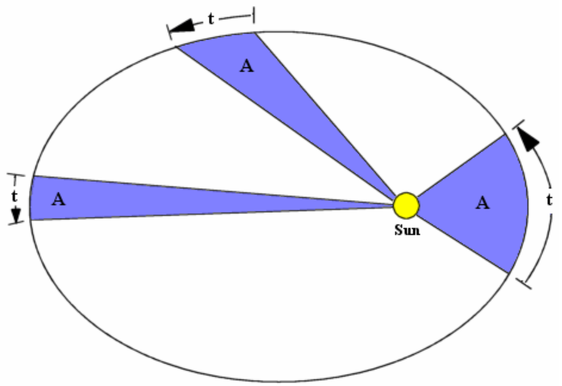
1st Law



✗

no elliptical orbits

2nd Law



theoretically: ✓

(conservation of momentum)

experiment: ✗

(energy losses – friction)

3rd Law

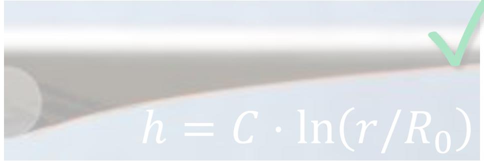
(for circular orbits)

$\cancel{\frac{a^3}{T^2}} = \text{const. for } \forall$ orbiting same mass

force equilibrium $(F_g = m\omega^2 a)$

$\frac{a^2}{T^2} \propto \frac{a}{T} = \text{const.}$ ✓

Thank you for your attention!



same motion ⇒

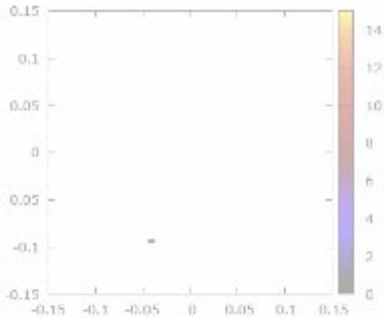
⇒ shape = potential

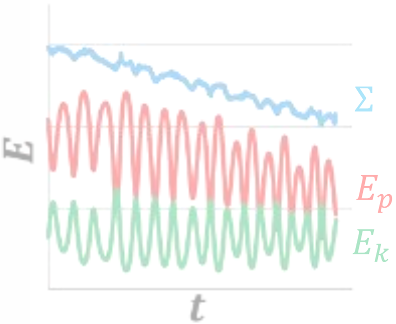
description of sheet curvature

same equations as gravity, but 2D

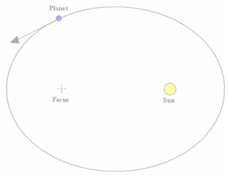
$$F = \kappa_m \frac{Mm}{r}; \quad \kappa_m = 0.053 kg^{-1} m^2 s^{-2}$$

dynamics:

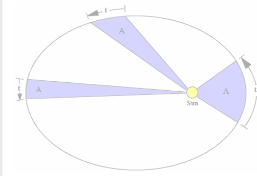




Kepler's Laws:



✗

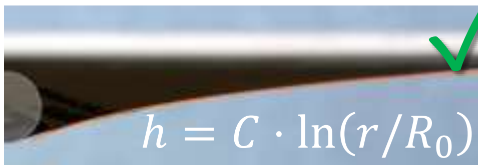


th. ✓, exp. ✗

$\cancel{\frac{a^3}{T^2} = C}$

$\frac{a}{T} = C$ ✓

Thank you for your attention!



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$$F = \kappa_m \frac{Mm}{r};\ \kappa_m = 0.053\,\frac{\mathrm{m}^2}{\mathrm{kg{\cdot}s}^2}$$

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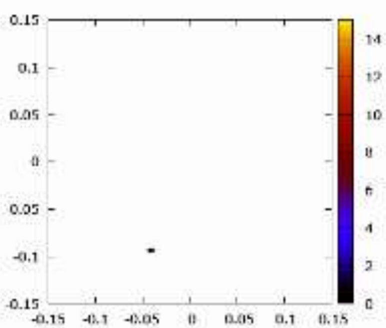
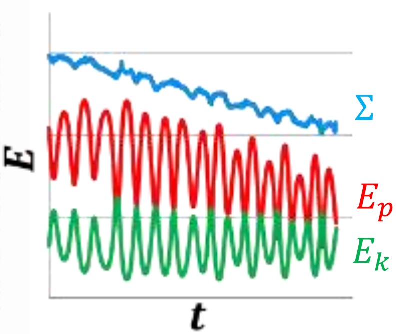
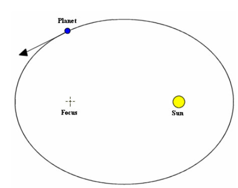
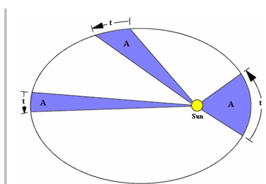




Kepler's Laws:





th.✓, exp.✗

$$\cancel{\frac{a^3}{T^2} = \mathrm{C}}$$

$$\frac{a}{T} = \mathrm{C}$$ ✓

APPENDIX

- gravity, rubber sheet: equations
- gravity in *N*-D
- derivation of Kepler's 3rd Law
- small slopes approximation

ϕ: Poisson's Equation

- intensity $\boldsymbol{g} = \frac{\boldsymbol{F}}{m}$: gradient of potential

$$\boldsymbol{g}(\boldsymbol{r}) = -\nabla\phi(\boldsymbol{r})$$

- Gauss's Theorem:

$$\nabla \cdot \boldsymbol{g}(\boldsymbol{r}) = -4\pi\kappa\rho(r)$$

- together: Poisson's equation

$$\nabla \cdot \left(-\nabla\phi(\boldsymbol{r})\right) = -4\pi\kappa\rho(r)$$

$$\boxed{\Delta\phi(\boldsymbol{r}) = 4\pi\kappa\rho(r)}$$

ϕ: Boundary Conditions, $\rho(\boldsymbol{r})$

$$\Delta\phi(\boldsymbol{r}) = 4\pi\kappa\rho(r)$$

- $\varphi(\infty) = \text{const.}$, therefore

$$\boxed{\partial_n\varphi|_S = 0}$$

- ρ: point mass in our measurements:

$$\boxed{\rho(\boldsymbol{r}) = m\,\delta(\boldsymbol{r}),}$$

where $\delta(\boldsymbol{r})$ is δ-function

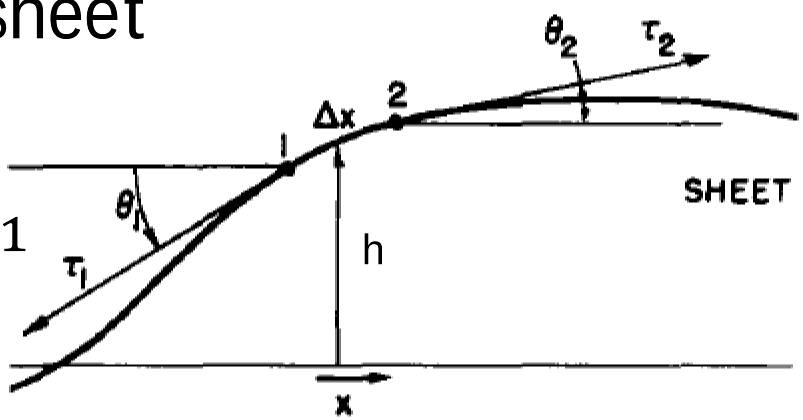
$\left(\int_{-\infty}^{\infty}\delta(x)\,\mathrm{d}x = 1,\ \forall x \in R - \{0\}: \delta(x) = 0\right)$

h: Poisson's Equation

- net force causing the rubber sheet to bend:

$$\Delta F = k\,\Delta y \sin\theta_2 - k\,\Delta y \sin\theta_1$$

$$\Delta F = k\Delta y(\sin\theta_2 - \sin\theta_1)$$



- for $\theta \ll 1$: $\sin\theta \approx \tan\theta \approx \frac{\partial h}{\partial x}$, then

$$\Delta F = k\,\Delta y\left(\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial x}\right) = \boxed{k\Delta y \frac{\partial^2 h}{\partial x^2}\,\Delta x}$$

k: "surface tension" (force per length)

h: Poisson's Equation

- deformation is caused by gravity:

$$\Delta F = g\,\rho \Delta x \Delta y$$

- substitute $\Delta F = k \Delta y \dfrac{\partial^2 h}{\partial x^2}\,\Delta x$:

$$\frac{\partial^2 h}{\partial x^2} = \frac{\Delta F}{k \Delta x\, \Delta y} = \frac{\rho g}{k}$$

- generalization (vector fields):

$$\boxed{\Delta h(\boldsymbol{r}) = \frac{g}{k}\sigma(\boldsymbol{r})}$$

h: Boundary Conditions, $\rho(\boldsymbol{r})$

$$\Delta\phi(\boldsymbol{r}) = \frac{g}{k}\rho(r)$$

- $\varphi(\infty) = \text{const.}$, therefore

$$\boxed{\partial_n h|_\infty = 0}$$

- ρ: area density negligible in comparison to mass of ball m, therefore approximated by δ-function:

$$\boxed{\rho(\boldsymbol{r}) = m\,\delta(\boldsymbol{r})}$$

Rubber Sheet vs. Gravity

- two differential equations of the same type, in the same region
  - von Neumann boundary conditions: $\frac{\partial_n f}{\partial r} = 0$
- in a closed region such differential equations have at most one solution (Dirichlet's problem, unique solution theorem)

→ character of solutions for ϕ and h will be the same – $\phi \propto h$

The Form of Newton's Law of Gravitation

- why $F_g \propto \frac{1}{r^2}$?
  - Gauss's Law: $\oiint_{\partial V} \boldsymbol{g} \cdot \mathrm{d}\boldsymbol{A} = -4\pi\kappa M$

    (*gravitational flux* through any closed surface is proportional to the enclosed mass)
  - special case: spherical symmetry, point mass:

$$g \cdot A_s = -4\pi\kappa M \; ; \; A_s = 4\pi r^2$$

$$\boxed{g = -\frac{4\pi\kappa M}{A_s}} = -\frac{\kappa M}{r^2}$$

The Form of Newton's Law of Gravitation

- why $F_g \propto \frac{1}{r^2}$?
  - Gauss's Law: $\oiint_{\partial V} \boldsymbol{g} \cdot \mathrm{d}\boldsymbol{A} = -4\pi\kappa M$

    (*gravitational flux* through any closed surface is proportional to the enclosed mass)
  - special case: spherical symmetry, point mass:
  - holds in *N* dimensions, too

$$g = -\frac{4\pi\kappa M}{A_s}\ ;\ A_s \propto r^{N-1} \Rightarrow \boxed{g \propto -\frac{\kappa M}{r^{(N-1)}}}$$

Gravitational Force in 2D

- $g = -\dfrac{2\kappa M}{r}$
- potential $\phi(\boldsymbol{r}) = \int_r^\infty \boldsymbol{g}\,\mathrm{d}\boldsymbol{r} = g\log r + \mathrm{C}$

Derivation of the 3rd Kepler's Law

$$F_g = F_{cf}$$

standard - 3D

$$\kappa_m \frac{Mm}{r^2} = m\omega^2 r$$

2D rubber sheet

$$\kappa_m \frac{Mm}{r} = m\omega^2 r$$

substitute $\omega = \frac{2\pi}{T}$:

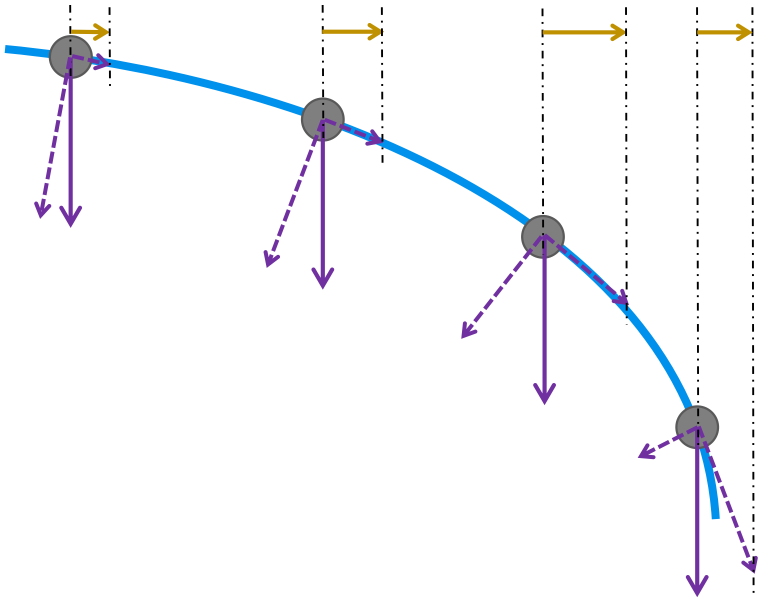
$$\frac{r^3}{T^2} = \frac{\kappa_m M}{4\pi^2}$$

$$\frac{r^2}{T^2} = \frac{\kappa_m M}{4\pi^2}$$

$$\boxed{\frac{r^3}{T^2} = \frac{1}{4\pi^2}\kappa M}$$

$$\boxed{\frac{r}{T} = \frac{1}{2\pi}\sqrt{\kappa_m M}}$$

Large slopes do not work well



if the slope is too big, the projected force will not be monotonous

+ part of E_k → vertical motion!

we will work only with small slopes

we will work only with small slopes

we will work only with small slopes

⇒ we can assume:

- uniform tension
- $\sin\theta \approx \theta \approx \tan\phi = \frac{\Delta h}{\Delta x}$
- $\cos\theta \approx 1$

- experiment:
  - no inelastic (permanent) deformation of membrane
  - Hooke's law holds (force $\propto$ deformation)