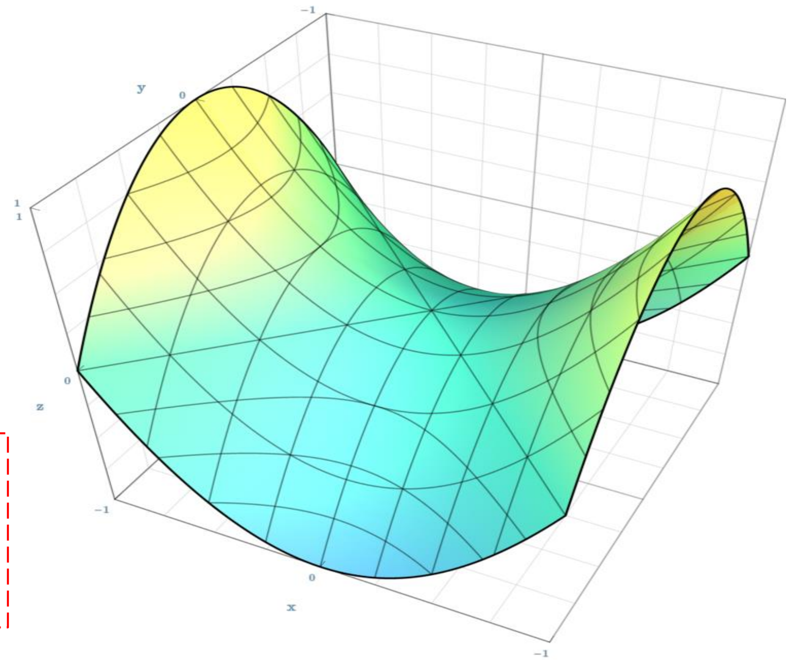
PROBLEM 13.

ROTATING SADDLE

Reporter: Domagoj Pluščec

Problem 13

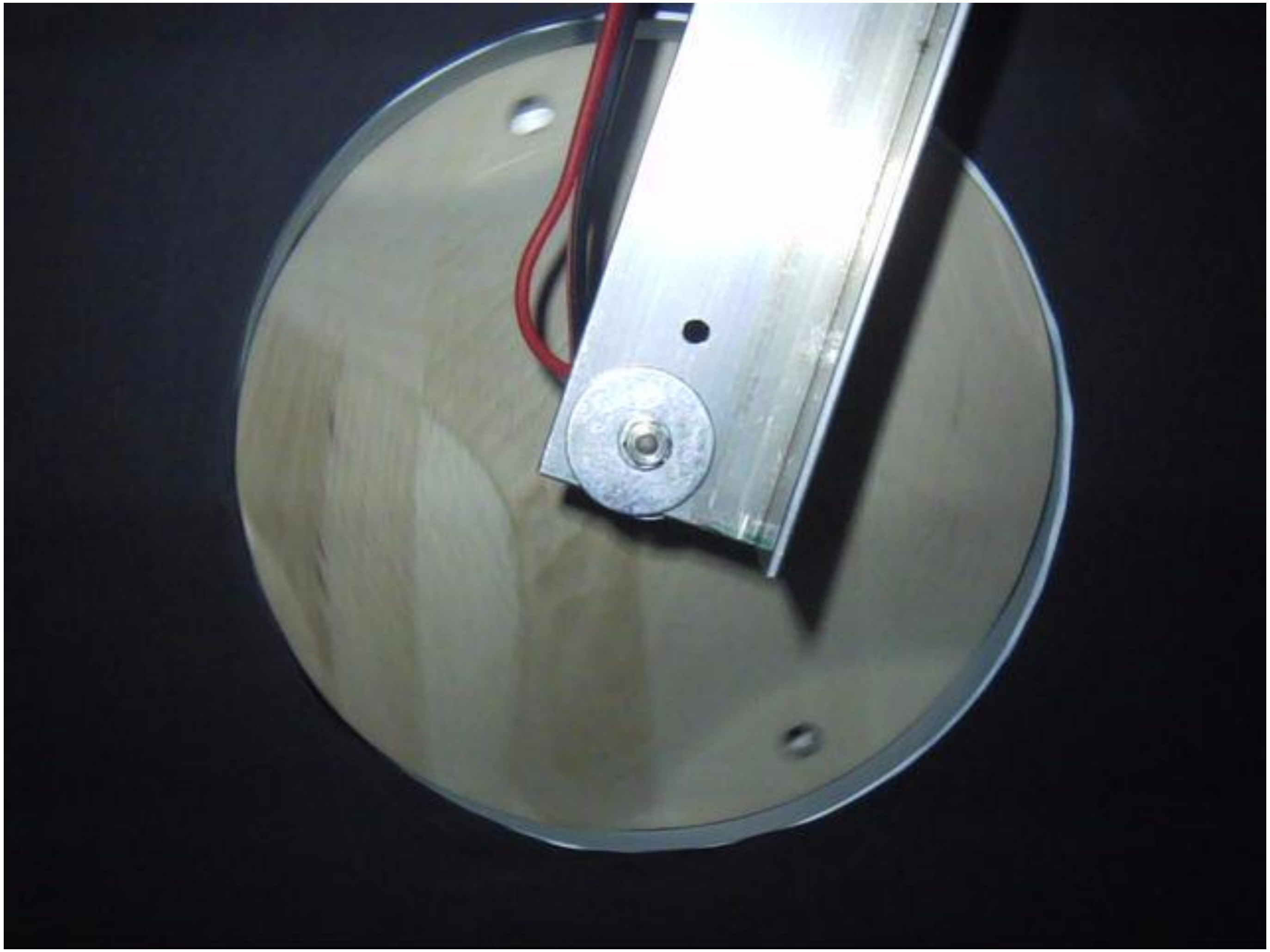
„A ball is placed in the middle of a rotating saddle. Investigate its dynamics and explain the conditions under which the ball does not fall off the saddle."



Saddle - hyperbolic paraboloid

$$z = \frac{x^2}{a} - \frac{y^2}{b}, \text{a}, \text{b} \in \mathcal{R}^+$$

Motion example



Outline

Theoretical modeling

- Analogy to Paul trap model
- Stability conditions
- Parameters

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Experimental aproach

- Saddle construction
- Ball releasing method
- Ball properties
- Determing rotational frequency
- Recording balls motion

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Results

- Ball trajectory
- Dependence of stabilization time on frequency
- Friction influence on stabilization

Theoretical modeling

- Paul trap analogy (R.I. Thompson, 2002)

Electric field potential → Gravitational potential

Alternating of electric potential → Rotation of the saddle

- Forces from rotational frame reference on the ball

$$m\vec{a} = \underbrace{-m\vec{\Omega} \times (\vec{\Omega} \times \vec{r})}_{\text{Centrifugal force}} - \underbrace{2\mathrm{m}\vec{\Omega} \times \vec{v}}_{\text{Coriolis force}} - \underbrace{\vec{\nabla} U}_{\text{Force caused by gravitational potential}} (*)$$

m – ball mass
Ω – saddle angular velocity
r – ball distance from center of the saddle

(*) as analogy we are neglecting friction, because it is difficult to solve analyticaly and we are going to disscus friction independentdtly

Gravitational potential of the saddle

$$U = mgz = mg\left(\frac{x^2}{a} - \frac{y^2}{b}\right)$$

$$U = \frac{m}{2}({\omega_1}^2 x^2 - {\omega_2}^2 y^2)$$

Supstitution:

$$\omega_1^2 = \frac{2g}{a}$$

$$\omega_2^2 = \frac{2g}{b}$$

- Assumptions:
  - If ball is stabilized it will be near the center of the saddle
  - Close to the center of the saddle we neglact the change in z direction $(\dot{y}, \dot{x} > \dot{z})$

- Using assumptions and projecting on (x,y) plane we obtain:

$$\ddot{x} - 2\Omega\dot{y} + ({\omega_1}^2 - \Omega^2)x = 0$$
$$\ddot{y} + 2\Omega\dot{x} - ({\omega_2}^2 + \Omega^2)y = 0$$

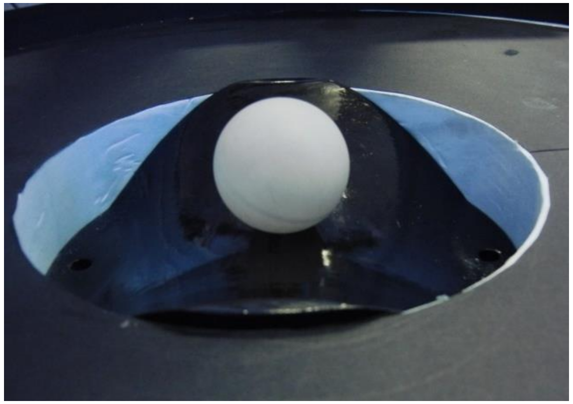
- For equations ansatz is in the form of $x = c_1 e^{-i\lambda t}$, $y = c_2 e^{-i\lambda t}$
  - All solutions for λ must be real for keeping ball stabilized because then solution of the equations is periodic function (Euler formula $e^{ix} = cosx + isinx$)
  - For lambda we obtain:

$$\lambda_{1,2}^2 = \frac{\omega_2^2 - \omega_1^2 - 2\Omega^2}{2} \pm \frac{1}{2}\sqrt{(\omega_1^2 + \omega_2^2)^2 + 8\Omega^2(\omega_1^2 - \omega_2^2)}$$

>0

>0

Stability conditions



- $\Omega = 0$ – point (0,0)
  - Unstable equilibrium $\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = 0$
  - In x direction stable equilibrium $\frac{\partial^2 U}{\partial x^2} > 0$
  - In y direction unstable equilibrium $\frac{\partial^2 U}{\partial y^2} < 0$

- $\Omega > 0$
  - ${\omega_2}^2 \leq {\omega_1}^2 \leq \Omega^2$

or

  - ${\omega_1}^2 \leq {\omega_2}^2 \leq 3{\omega_1}^2$(*) and ${\omega_1}^2 \leq \Omega^2 \leq \frac{({\omega_1}^2+{\omega_2}^2)^2}{8({\omega_1}^2-{\omega_2}^2)}$

Supstitution:

$$\omega_1^2 = \frac{2g}{a}$$

$$\omega_2^2 = \frac{2g}{b}$$

(*)obtained by investigating saddle curvature properties - Oleg N. Kirillov, 2010

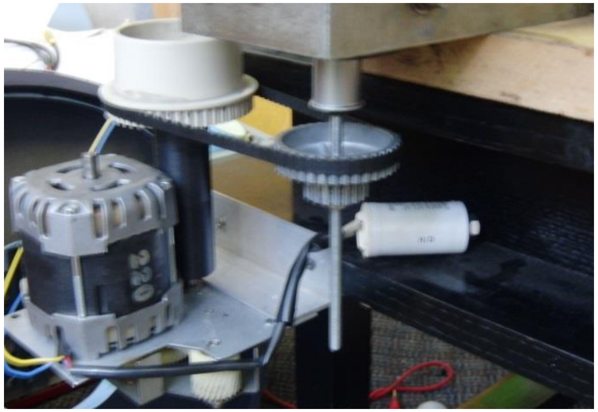
Parameters

- Saddle shape
- Rotating frequency
- Friction impact
- Moment of inertia of the ball (influence of the rotation of the ball)

Experiment

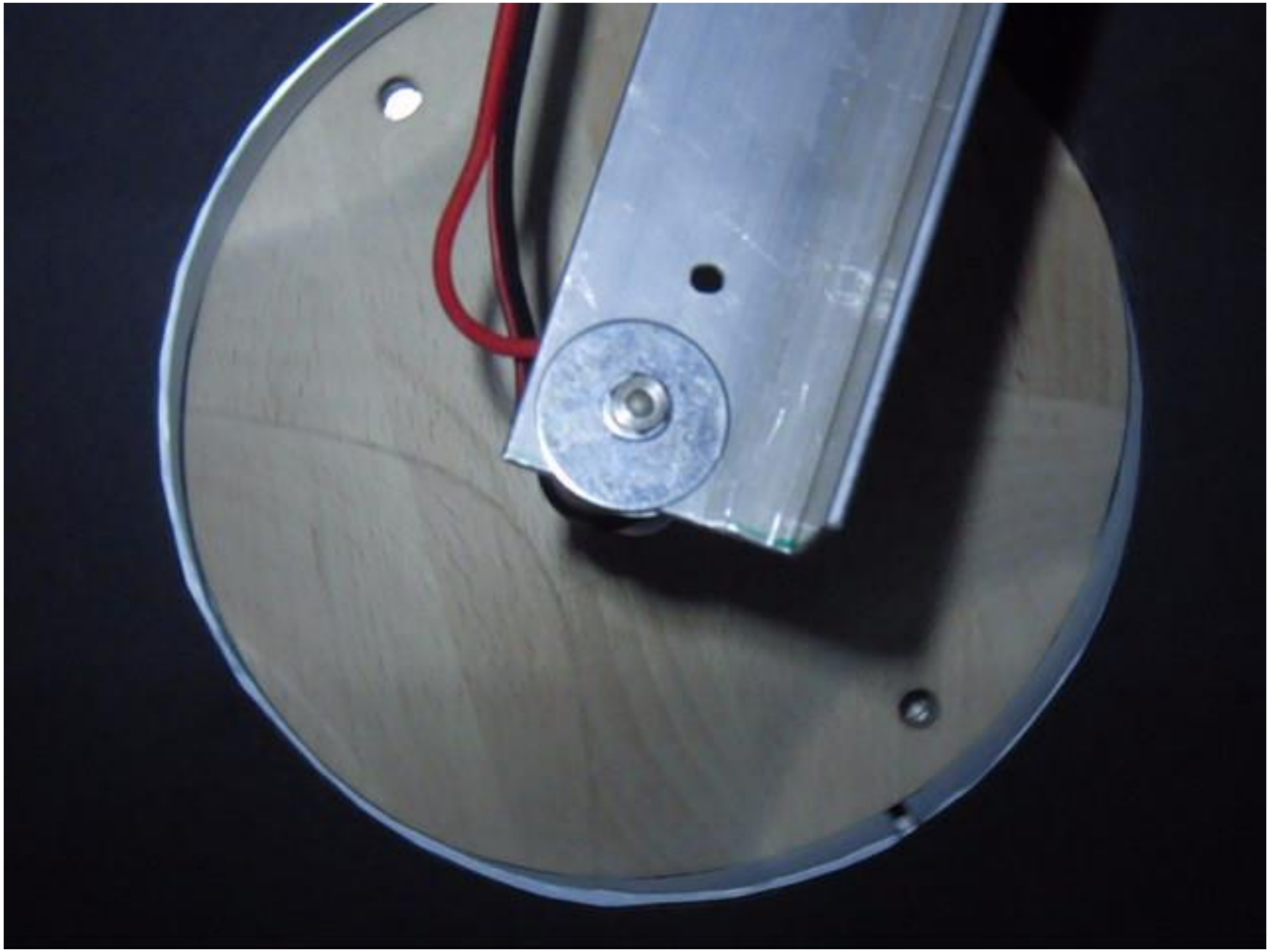
- Saddle construction
  - Saddle made by CNC miling machine
  - Saddle $z = \frac{x^2}{21} - \frac{y^2}{21}$ $[\mathrm{cm}]$, saddle radius 7.5 cm
- Ball releasing
  - Metal balls
    - Released by a coil (interuption of current caused ball drop)
    - Balls with radius 8.5 mm and 12.5 mm
  - Plastic and rubber balls
    - Released by hand
    - Balls with radius: 13 mm, 20 mm, 37.5 mm

Experiment



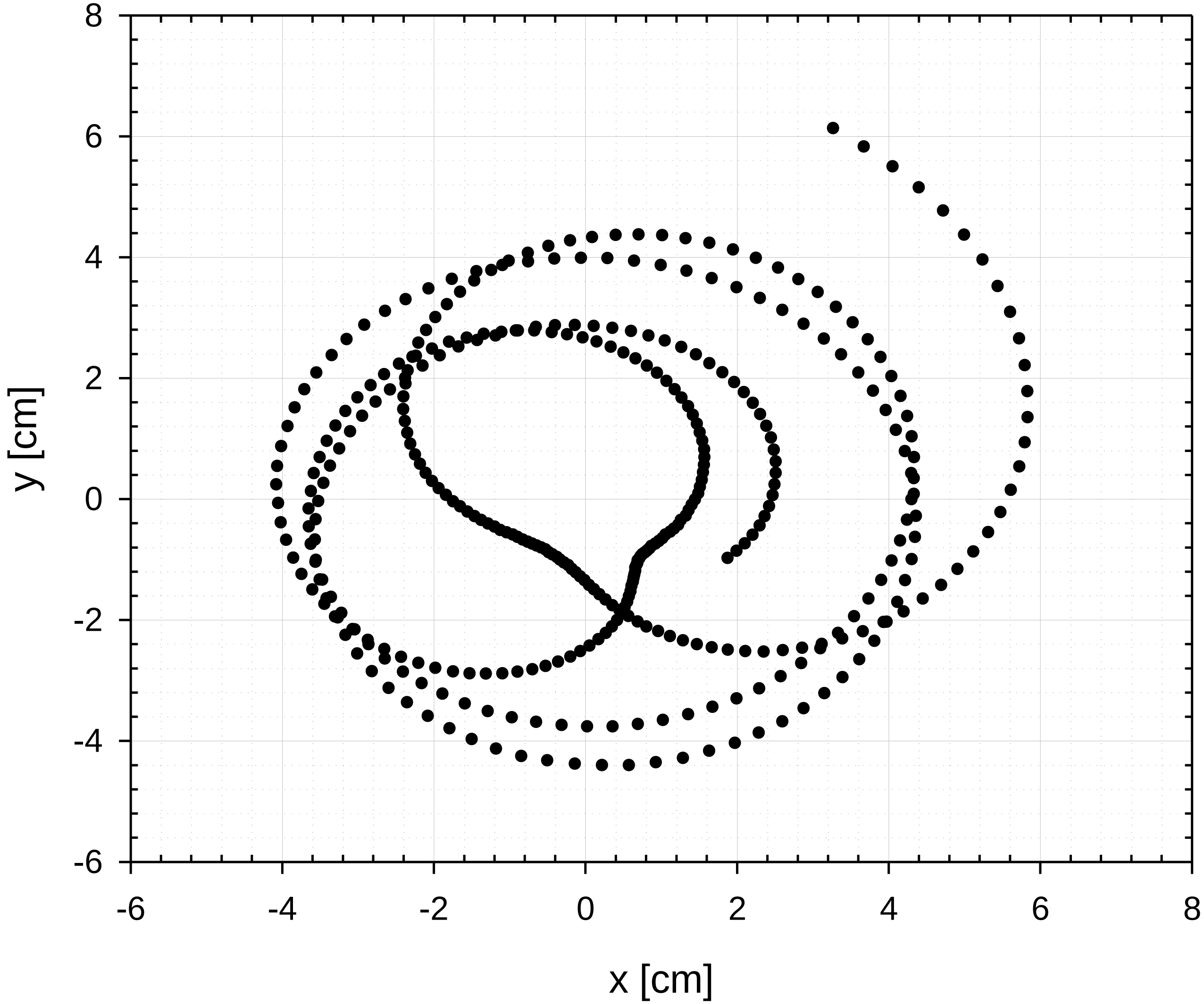
- Determing frequency of saddle rotation
  - Determined with stroboscope
  - Frequency of rotation of constructed turntable in range from 1 Hz to 5.5 Hz

- Determing ball motion
  - Motion recorded with high speed camera (120 fps)
  - Tracked in program for video analysis ImageJ
    - Center and axis of symetry of the saddle were determined by marks on the saddle

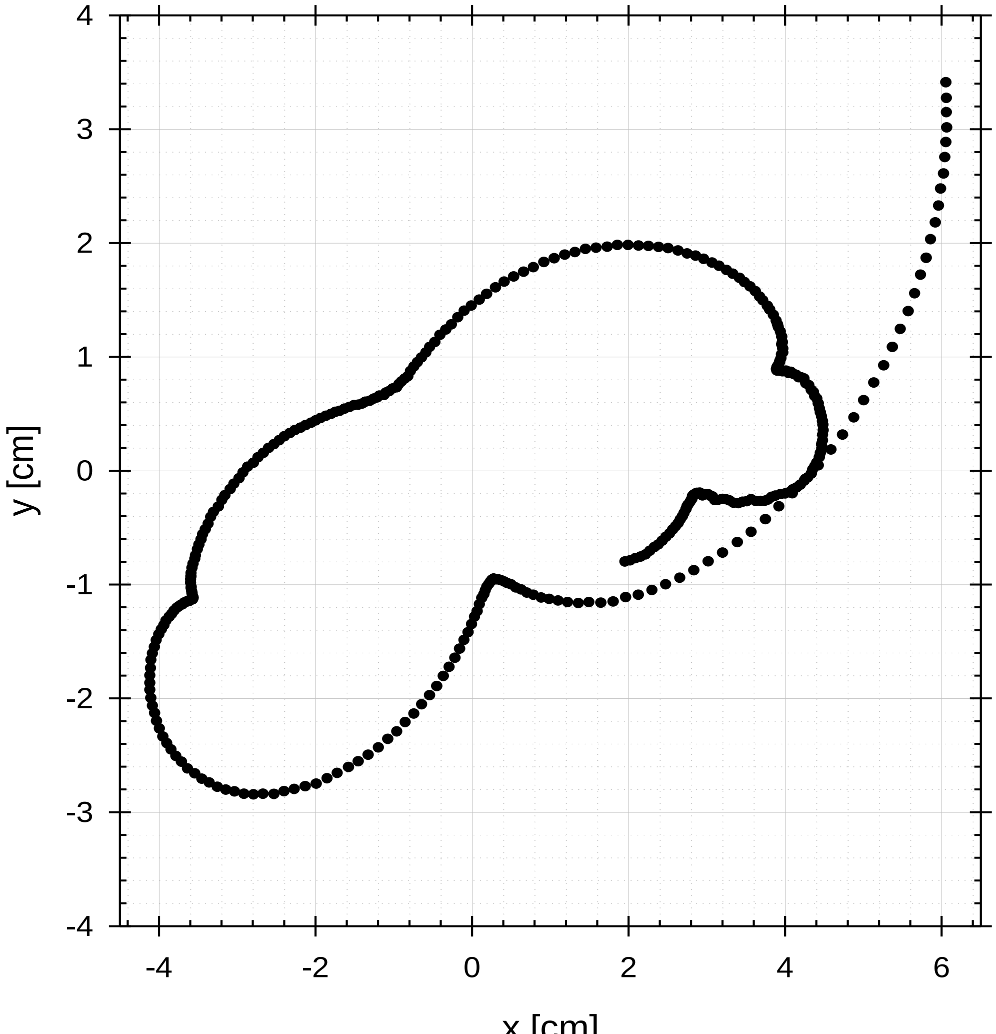


$f = 1.71\ Hz$
Metal ball
$r = 8.5$ mm

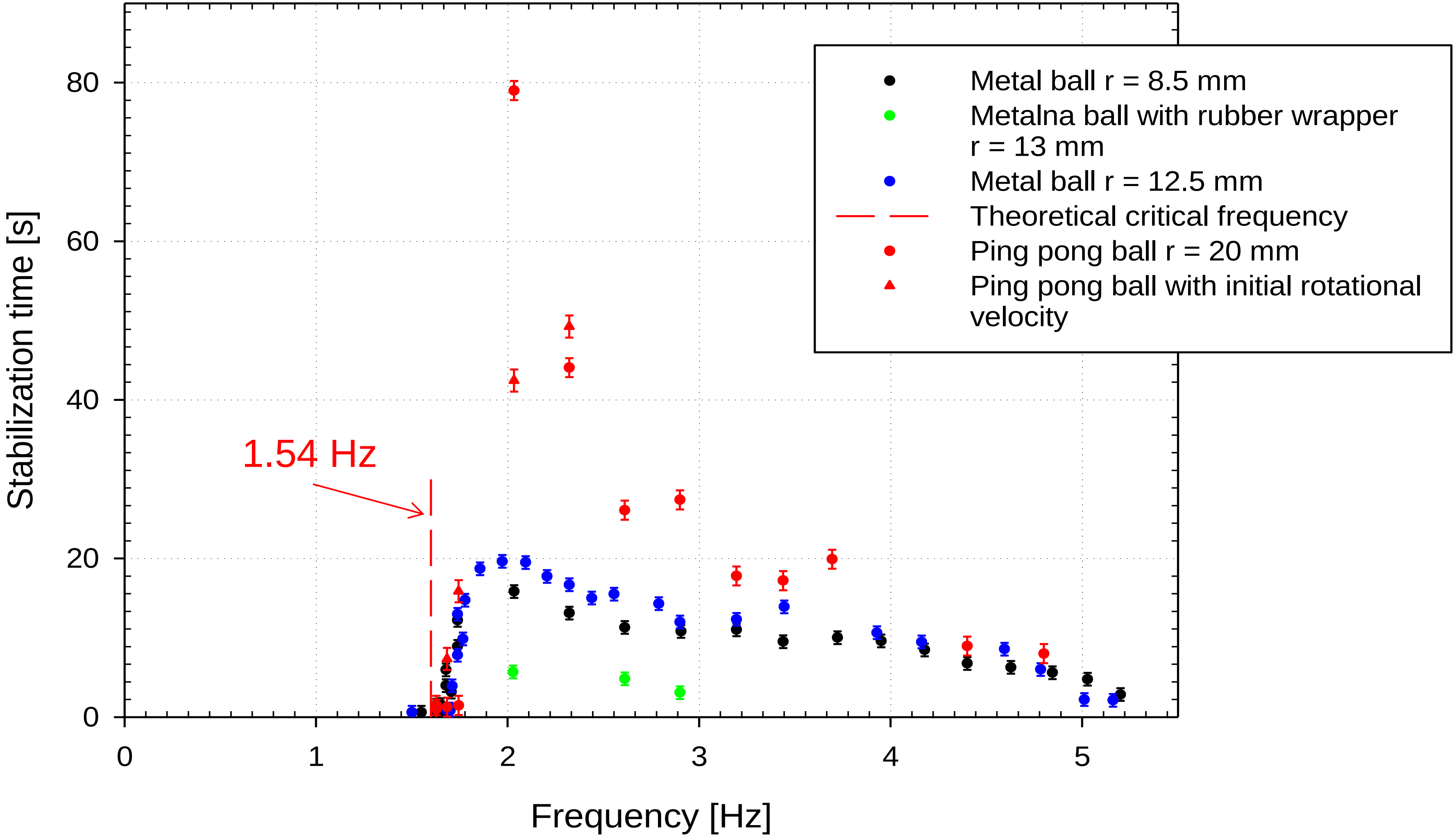
Ball trajectory - labaratory frame reference



Ball trajectory - rotational frame reference



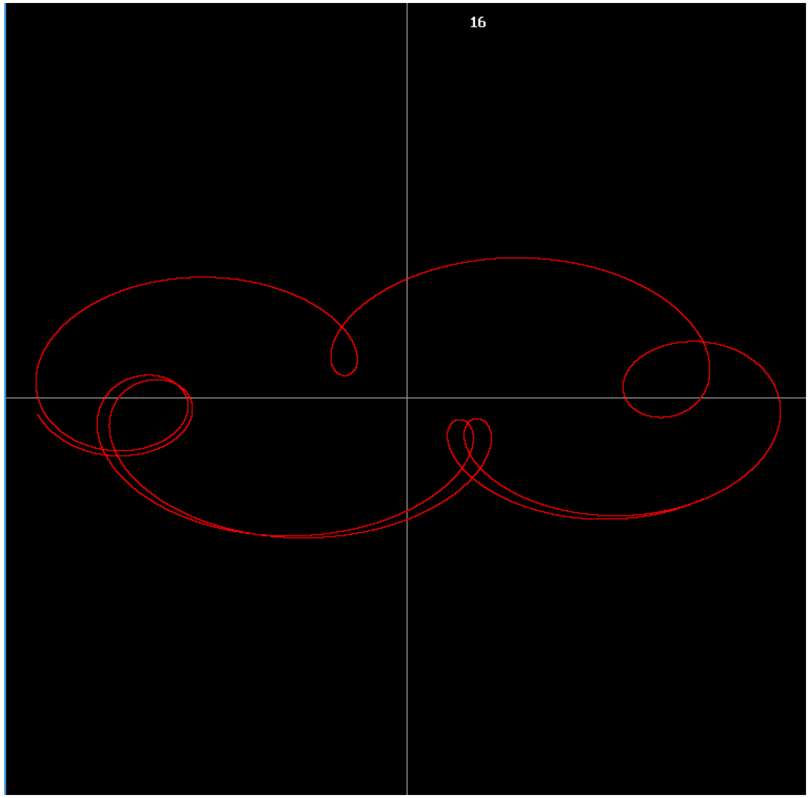
Dependence of stabilization time on saddle frequency



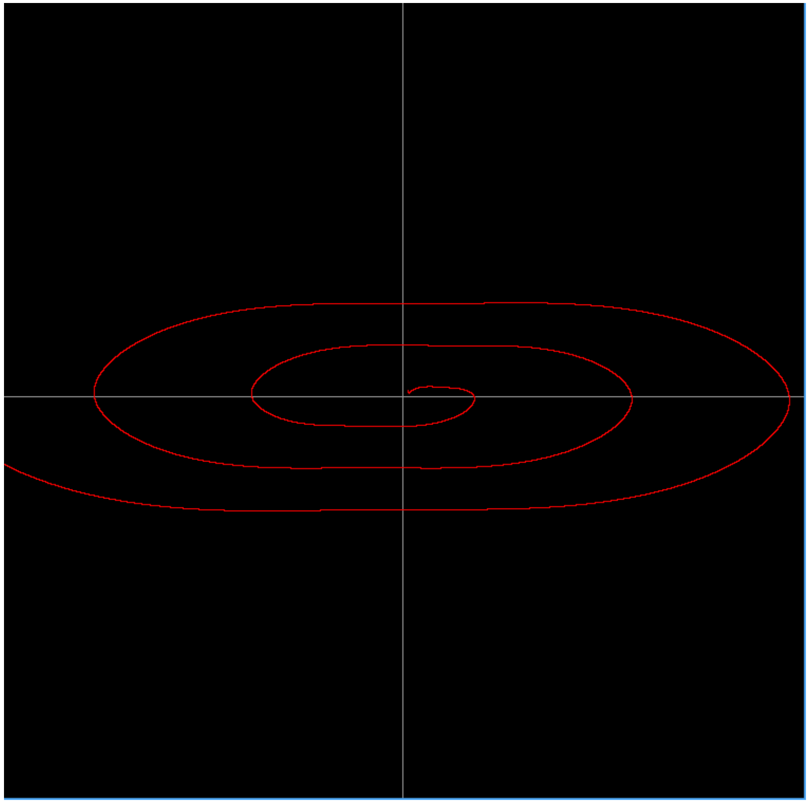
Friction impact – theoretical analysis

- We assume that when the ball is moving on the saddle that it is rolling
- For ball near the center we neglect it's movement in z direction ($\dot{y}, \dot{x} > \dot{z}$)
- We aproximated friction force in forme of $\vec{F} = \mu m g \frac{\vec{v}}{|v|}$ where μ is coefficient of rolling friction
- $\mu \sim 0.001$*
- We solved numericaly equations from our model and numericaly added friction force and observed how it influences our model

*Physics and Chemistry of Interfaces; Hans-Jürgen Butt, Karlheinz Graf, 2003



$\mu = 0; \mathrm{f} = 1.6$ Hz
picture dimensions 0.9x0.9 cm



$\mu = 0.005; \mathrm{f} = 1.6$ Hz
picture dimensions 15x15 cm

Conclusion

- We constructed a saddle and experimental setup for rotation of a saddle and system for releasing metal balls

- Simple theoretical model for point mass
  - Stabilization conditions:
    - $\omega_2{}^2 \leq \omega_1{}^2 \leq \Omega^2$ or
    - $\omega_1{}^2 \leq \omega_2{}^2 \leq 3\omega_1{}^2$ and $\omega_1{}^2 \leq \Omega^2 \leq \frac{(\omega_1{}^2+\omega_2{}^2)^2}{8(\omega_1{}^2-\omega_2{}^2)}$
  - Assumption:
    - $\dot{y}, \dot{x} > \dot{z}$ for the ball near the center of the saddle

- We discussed friction impact on ball stability
  - Friction will always destabilises ball on the rotating saddle

Literatura

1. Johan Nilsson; Trapping massless Dirac particles in a rotating saddle; Phys. Rev. Lett. 111, 100403 (2013)
2. R.I.Thompson, T.J. Harmon, M.G. Ball; The rotating saddle trap: a mechanical analogy to RF-electric-quadrupole ion trapping?; Can. J. Phys. 80, 1433-1448 (2002)
3. Oleg N. Kirillov, Brouwer's problem on a heavy particle in a rotating vessel: wave propagation, ion traps, and rotor dynamics; Physics Letters A 375, 1653-1660 (2011)
4. Physics and Chemistry of Interfaces. Hans-Jürgen Butt, Karlheinz Graf, Michael Kappl, 2003

Stroboscopic picture of the balls motion

IYPT 2014
CROATIAN TEAM

THANK YOU

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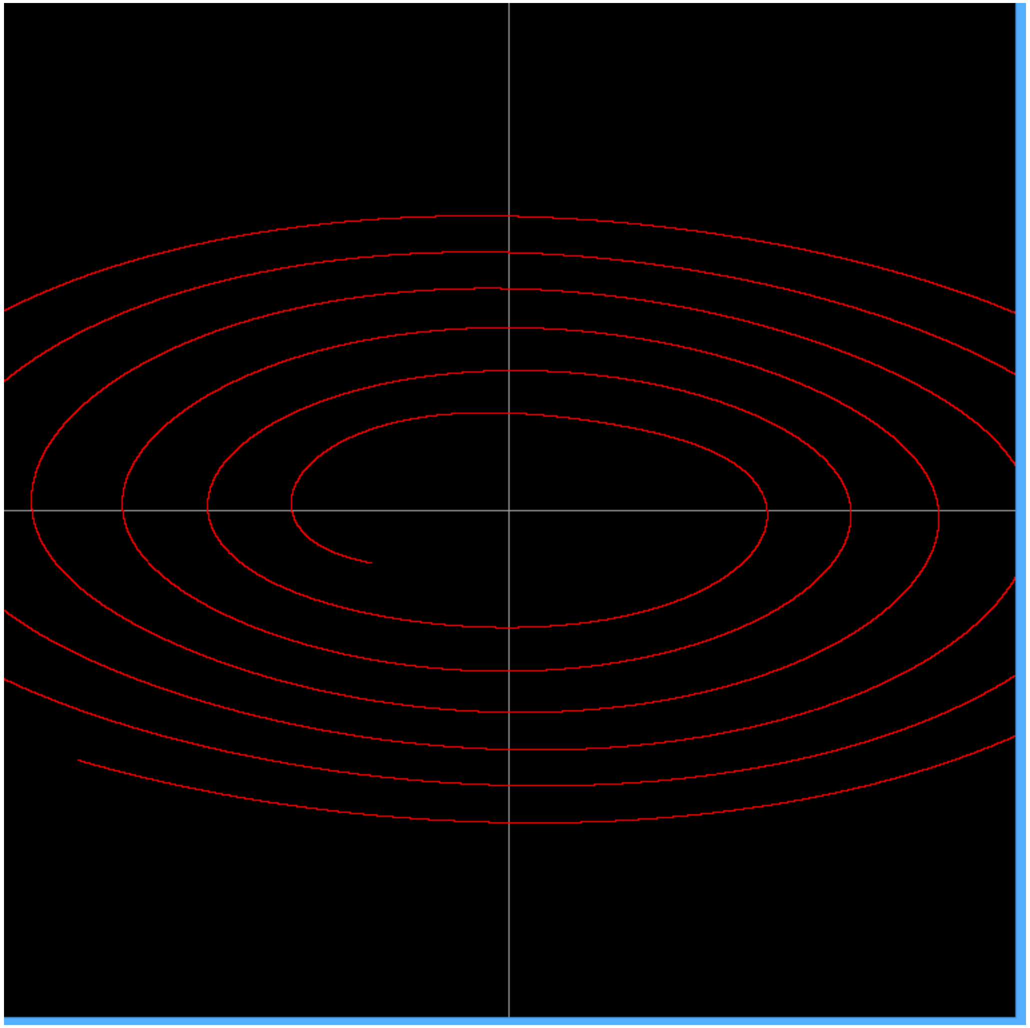
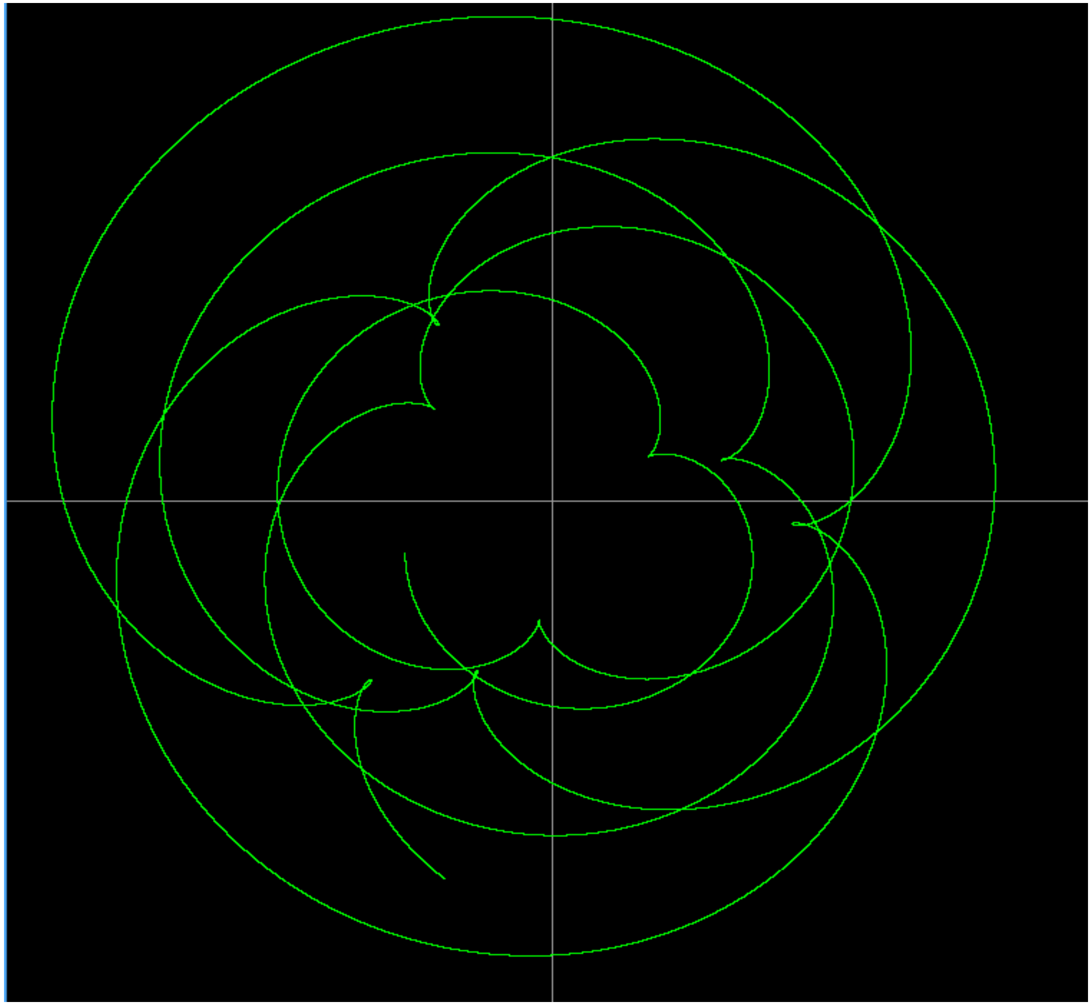
Ball trajectory – rotational frame reference

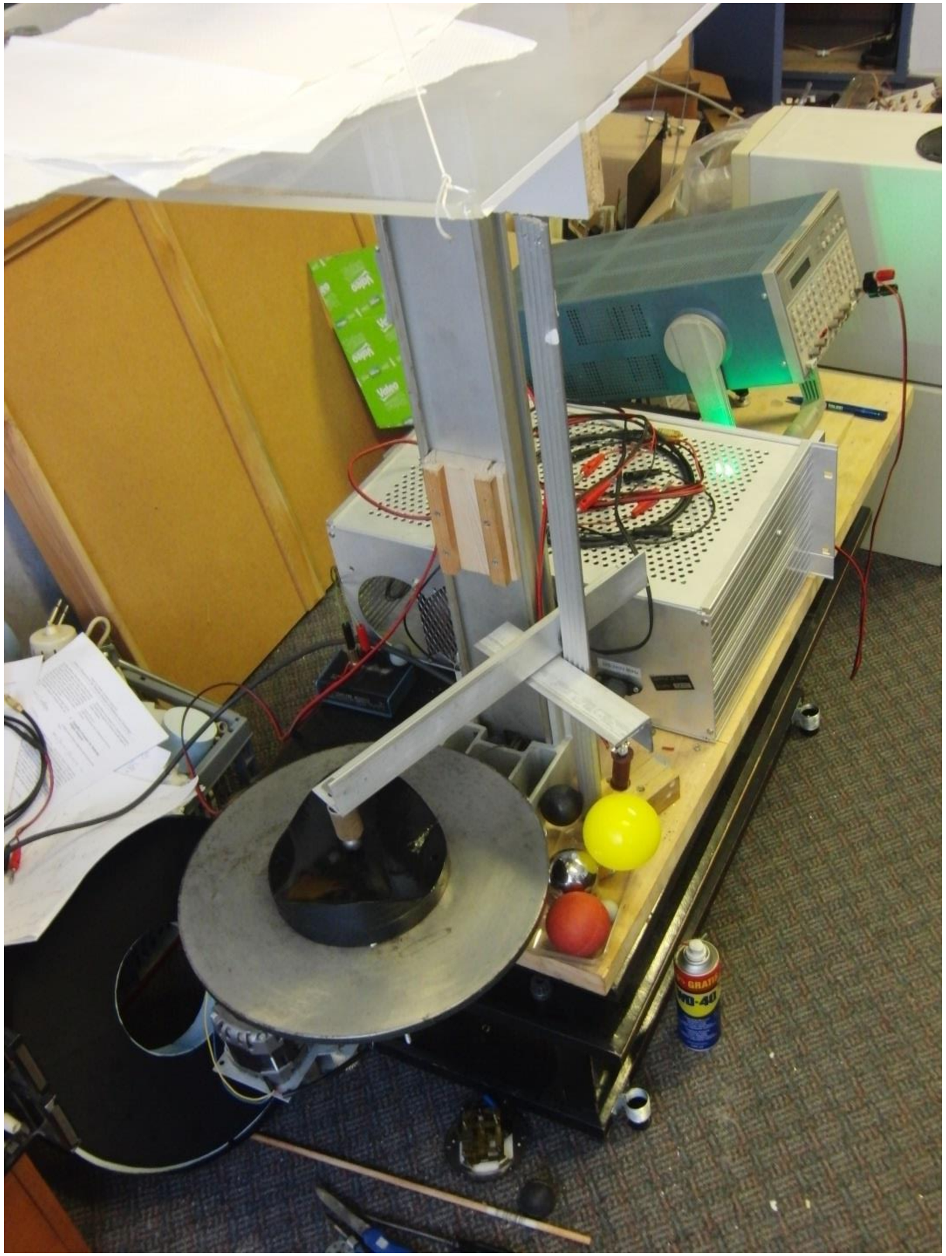


$\mu = 0.005; f = 1.71\ Hz$

Ball trajectory - labaratory frame reference



$\mu = 0.005; f = 1.71\ Hz$



$$\ddot{x} - 2\Omega\dot{y} + ({\omega_1}^2 - \Omega^2)x = 0$$
$$\ddot{y} + 2\Omega\dot{x} - ({\omega_2}^2 + \Omega^2)y = 0$$

$$-\lambda^2 c_1 + 2c_2\lambda\Omega i + ({\omega_1}^2 - \Omega^2)c_2 = 0$$
$$\lambda^2 c_2 + 2c_1\lambda\Omega i + ({\omega_2}^2 + \Omega^2)c_2 = 0$$

Det=0

$$\lambda^4 + (\omega_2^2 - \omega_1^2 - 2\Omega^2)\lambda^2 - (\omega_1^2 - \Omega^2)(\omega_2^2 + \Omega^2) = 0$$

$$\lambda_{1,2}^2 = \frac{\omega_2^2 - \omega_1^2 - 2\Omega^2}{2} \pm \frac{1}{2}\sqrt{(\omega_1^2 + \omega_2^2)^2 + 8\Omega^2(\omega_1^2 - \omega_2^2)}$$