02

Lagging Pendulum

Martin Gažo

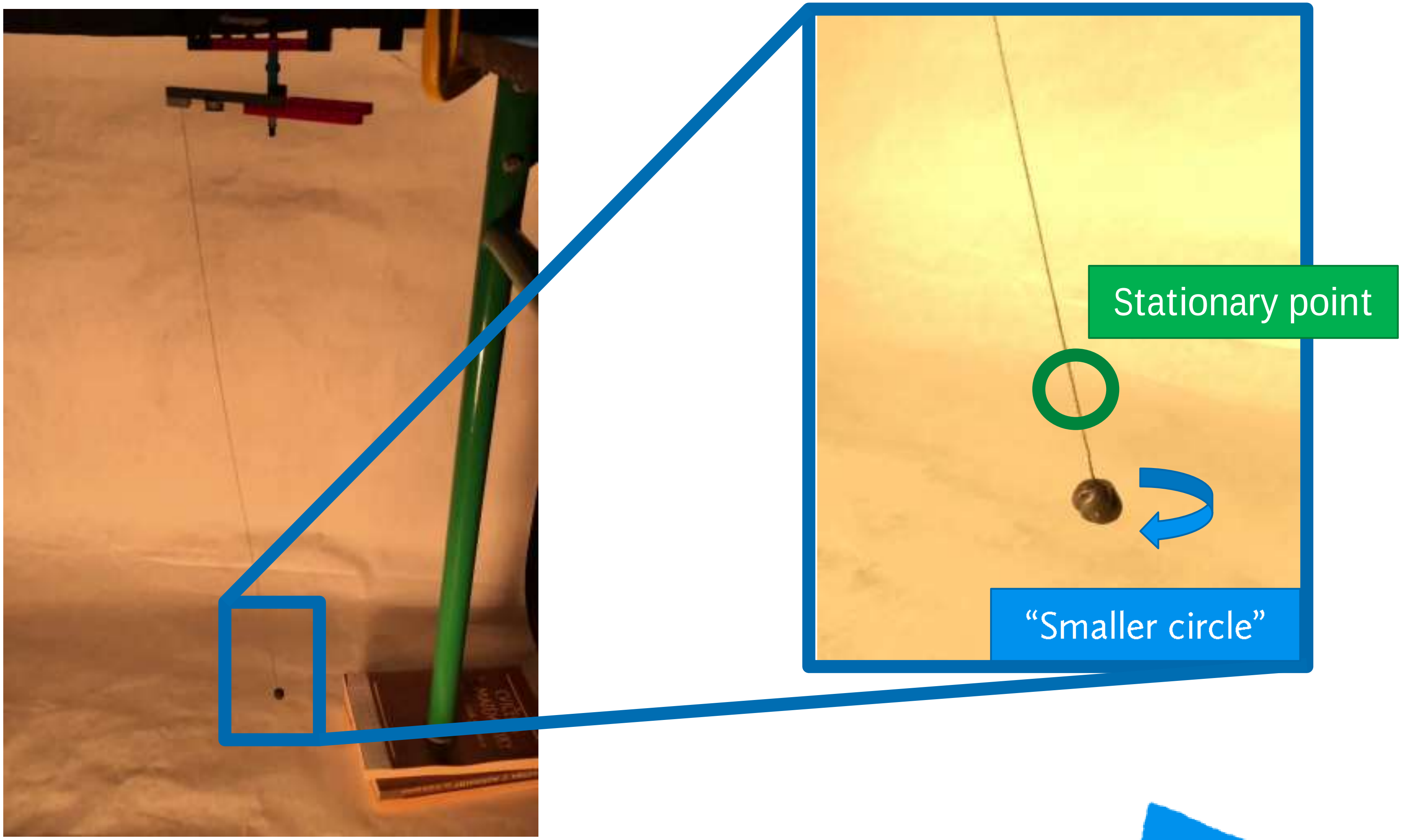
Task

A pendulum consists of a strong thread and a bob.

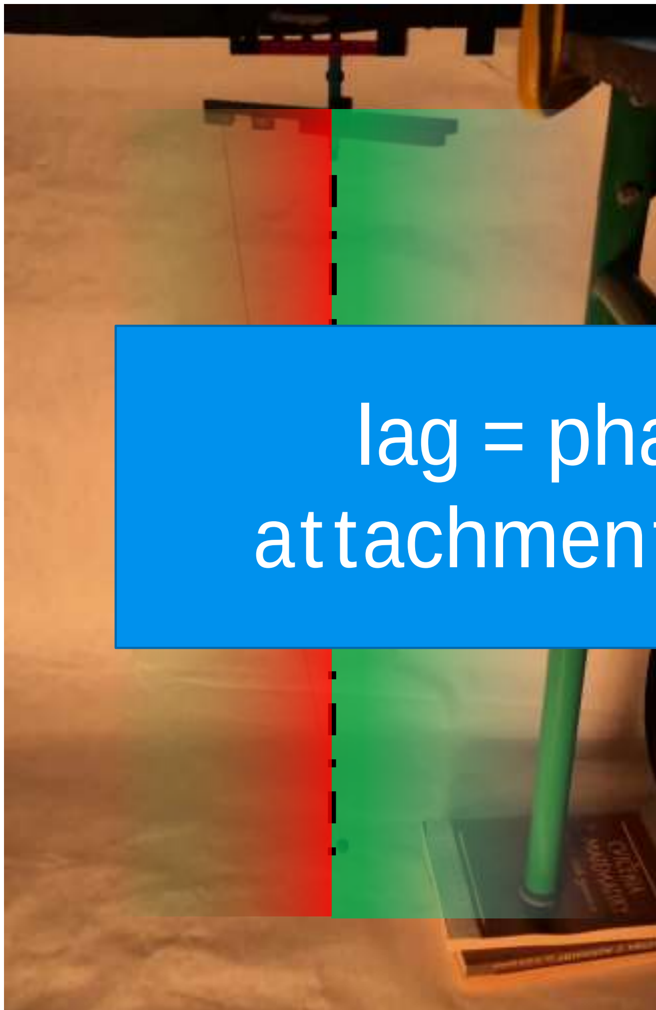
When the pivot of the pendulum starts moving along a horizontal circumference, the bob starts tracing a circle which can have a smaller radius, under certain conditions.

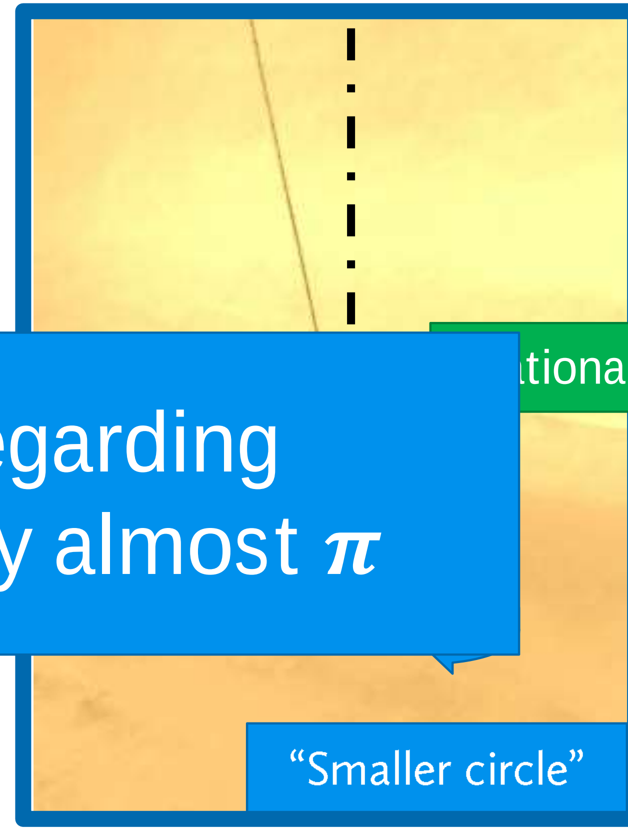
Investigate the motion and stable trajectories of the bob.

What does it look like?



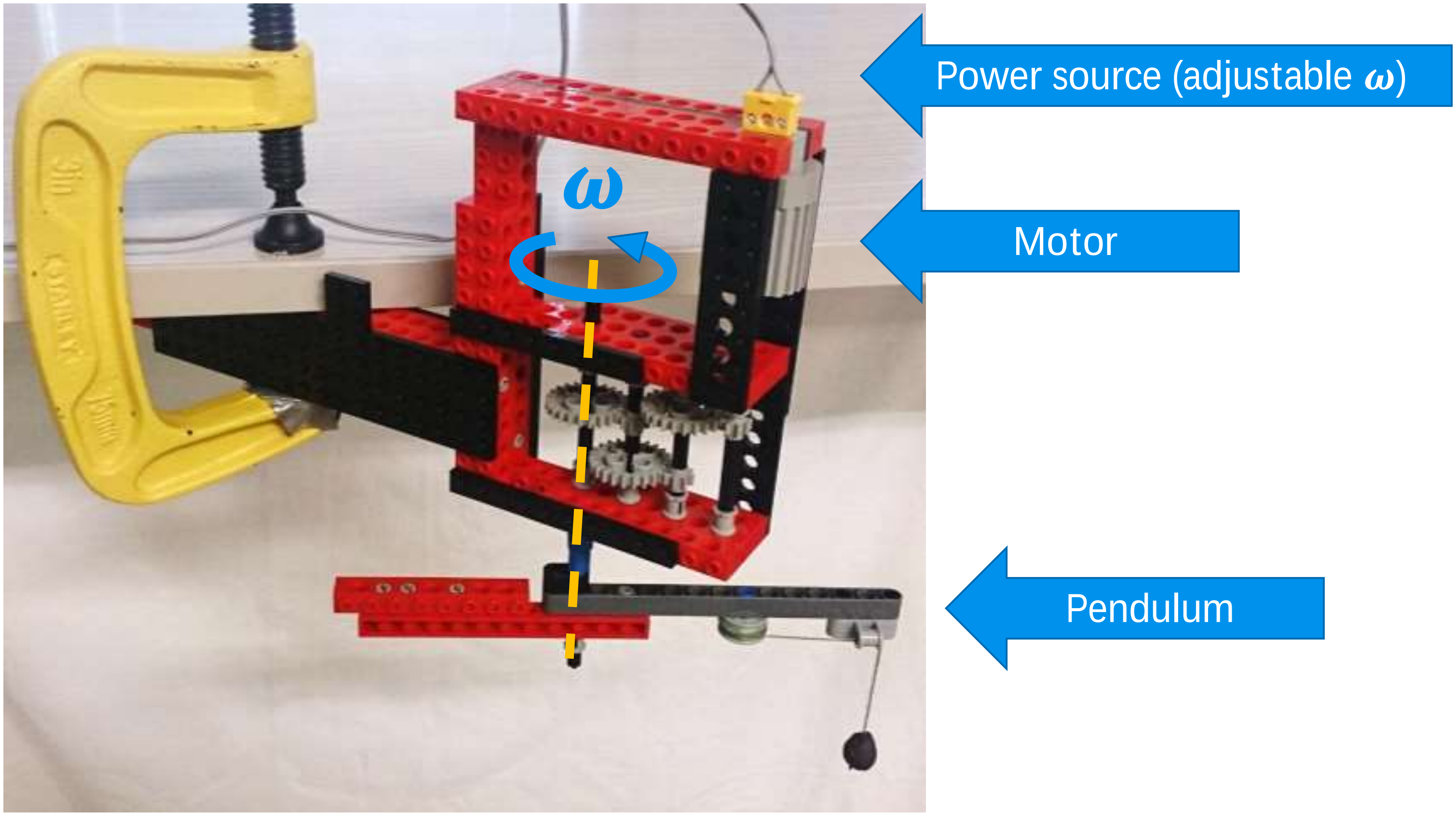
Lagging pendulum



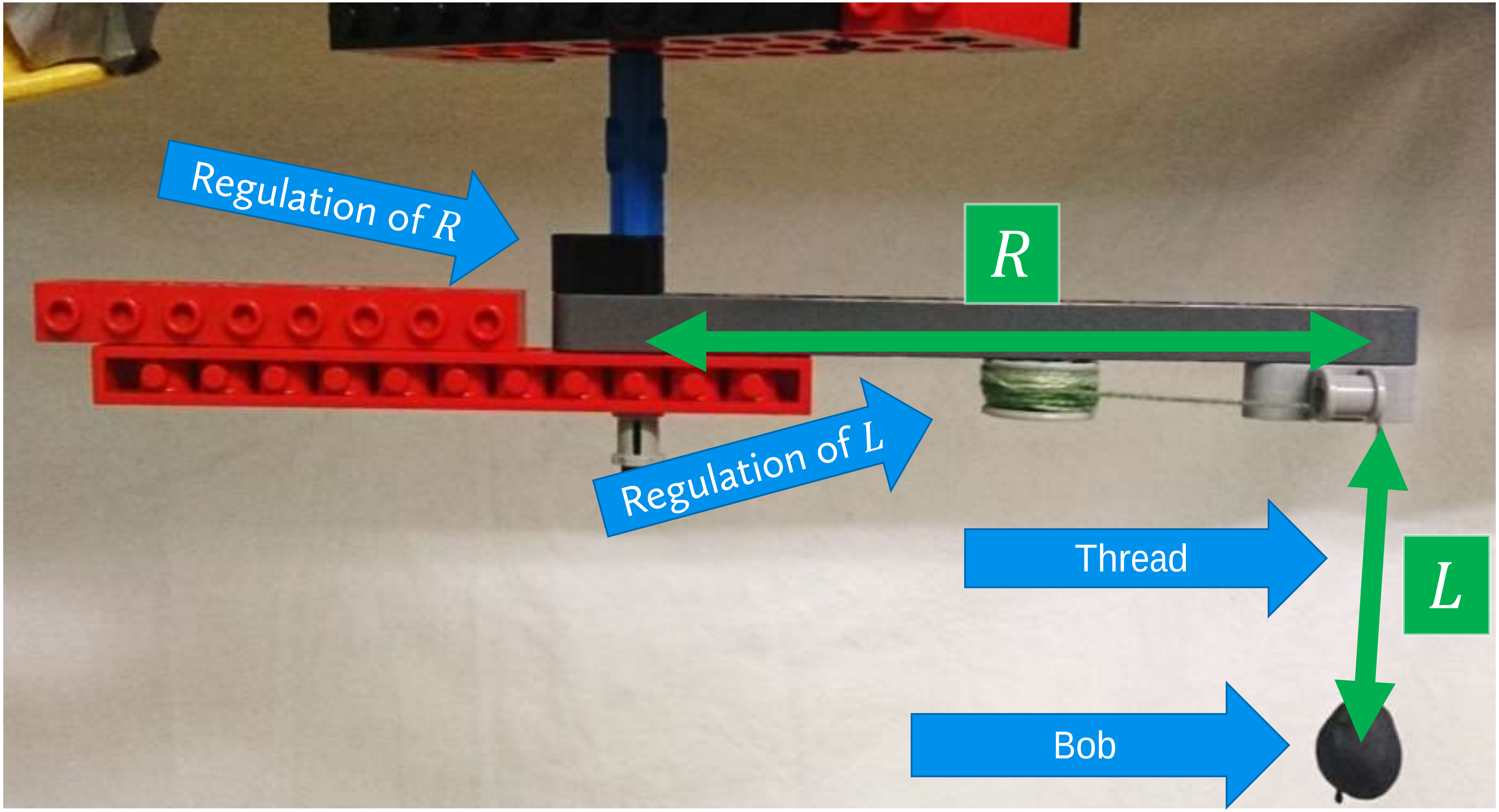


lag = phase lag regarding attachment point by almost π

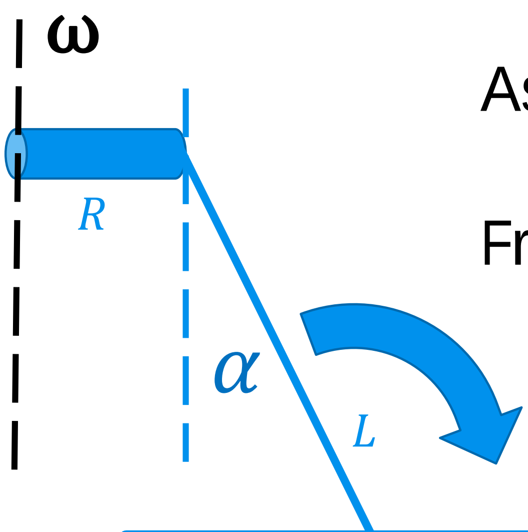
Apparatus



Apparatus



Analytical theory: circular trajectories



Assumption: bob moves only in vertical plane

Frame of reference: rotating with the arm

Fo
th

Angular acceleration:

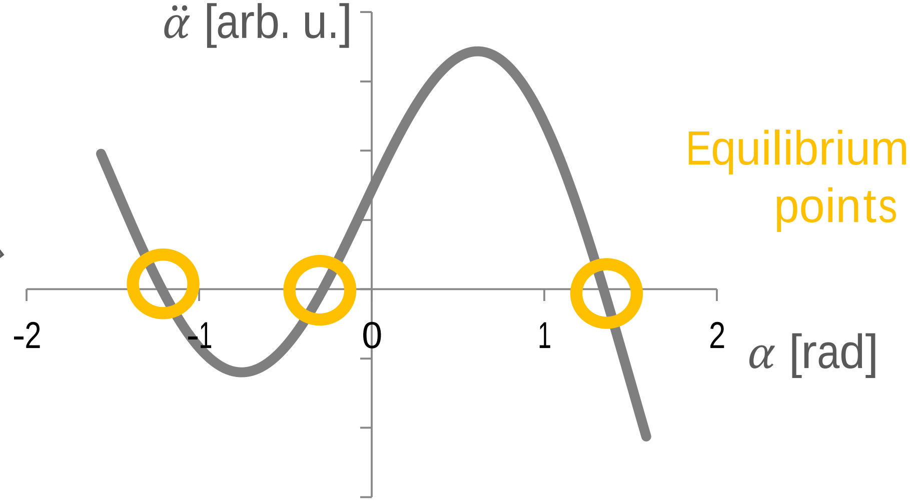
$$\omega_0 = \sqrt{g/L}$$

$$\ddot{\alpha}(\alpha) = \underbrace{-\omega_0^2 \sin\alpha}_{\text{Gravity}} + \underbrace{\omega^2 \left(\frac{R}{L} + \sin\alpha\right) \cos\alpha}_{\text{Centrifugal}}$$

G
force

Analytical theory: circular trajectories

What's the typical dependence of $\ddot{\alpha}$ on α?



$$\omega_0 = \sqrt{g/L}$$

$$\ddot{\alpha}(\alpha) = -\omega_0^2 \sin\alpha + \omega^2 \left(\frac{R}{L} + \sin\alpha\right) \cos\alpha \ = \mathbf{0}$$

Equilibrium condition

Analytical theory: circular trajectories

$$\omega_0 = \sqrt{g/L}$$

$$\ddot{\alpha}(\alpha) = -\omega_0^2 \sin\alpha + \omega^2 \left(\frac{R}{L} + \sin\alpha\right)\cos\alpha \quad = \mathbf{0}$$

Equilibrium condition

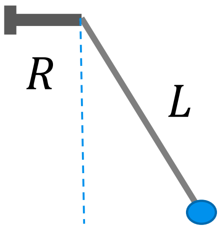
Analytical theory: circular trajectories

Equlibrium condition:

$$-\sin\alpha + \frac{\omega^2}{\omega_0^2}\left(\frac{R}{L} + \sin\alpha\right)\cos\alpha = \mathbf{0}$$

$$\omega_0 = \sqrt{g/L}$$

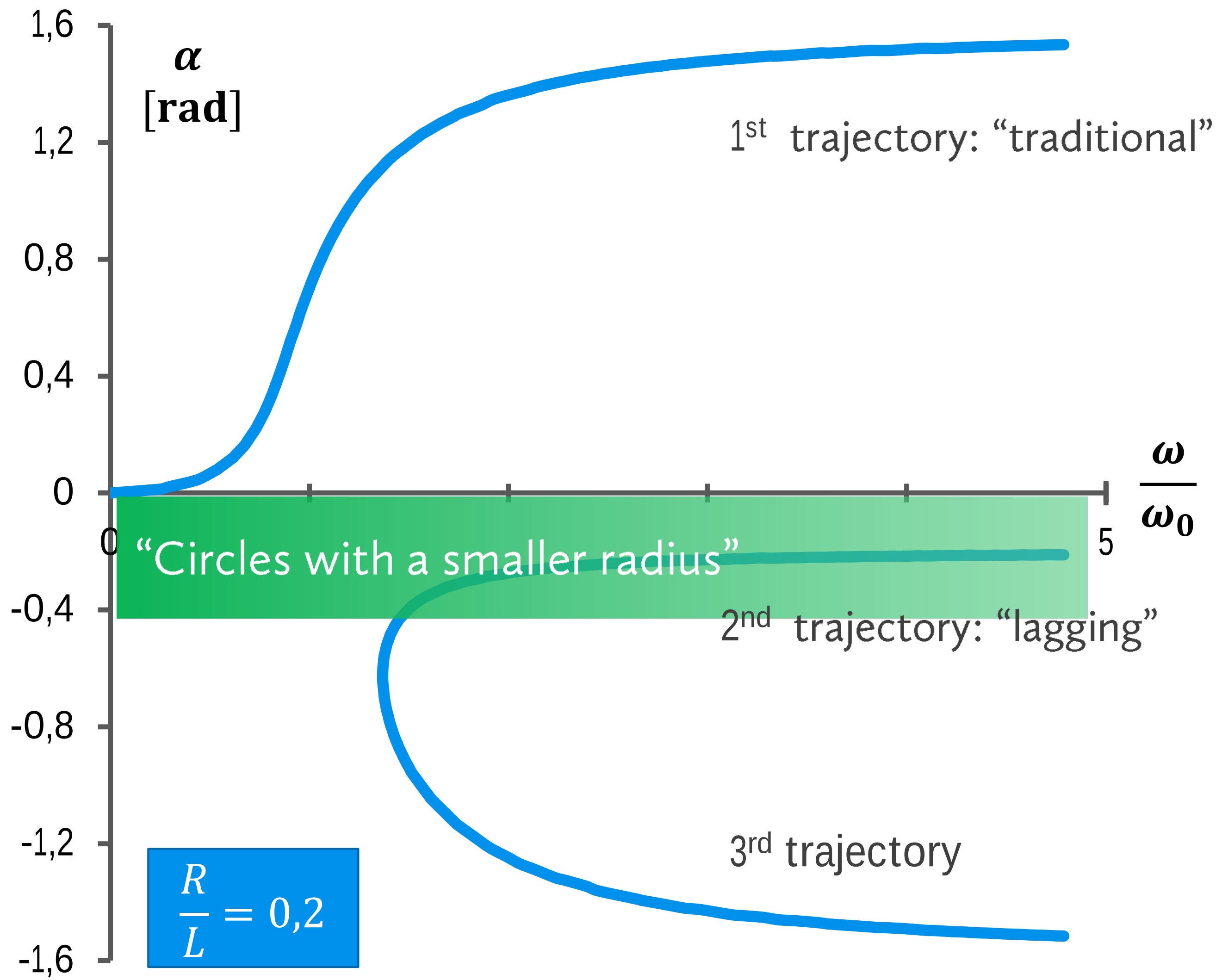
Two free parameters: ω/ω_0 and R/L



Natural frequency
(Depends on L)

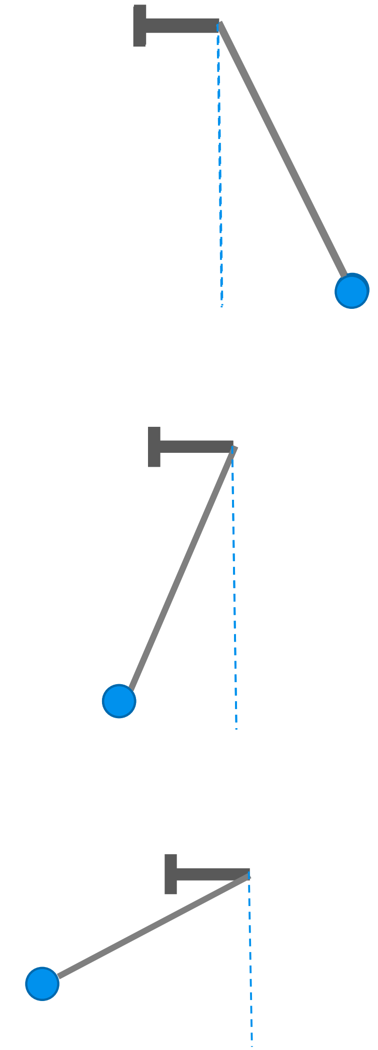
Let's plot solution in the dependence of ω/ω_0

Equilibrium angles

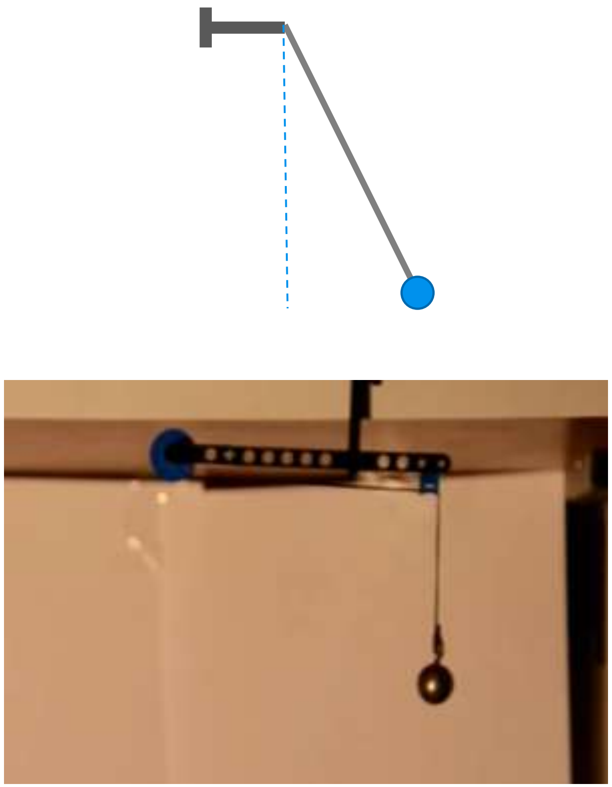


$\frac{R}{L} = 0{,}2$

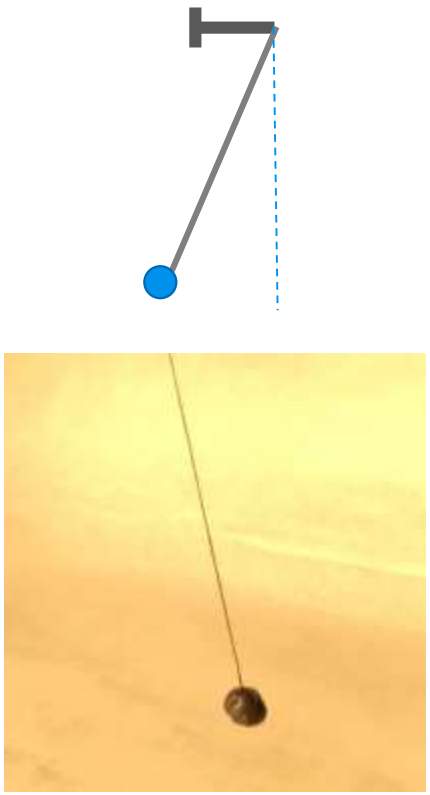
$\omega_0 = \sqrt{g/L}$



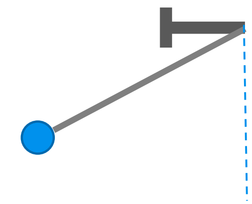
Spotted circular trajectories



✓ Observed

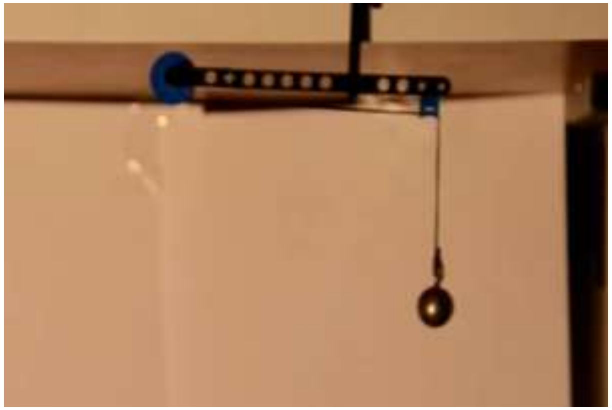


✓ Observed



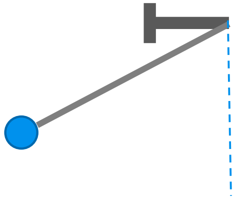
✗ Not observed

Spotted circular trajectories



✓ Observed

✓ Observed



✗ Not observed

Possible flaw: equilibrium doesn't necessary imply stability

Are the trajectories stable?

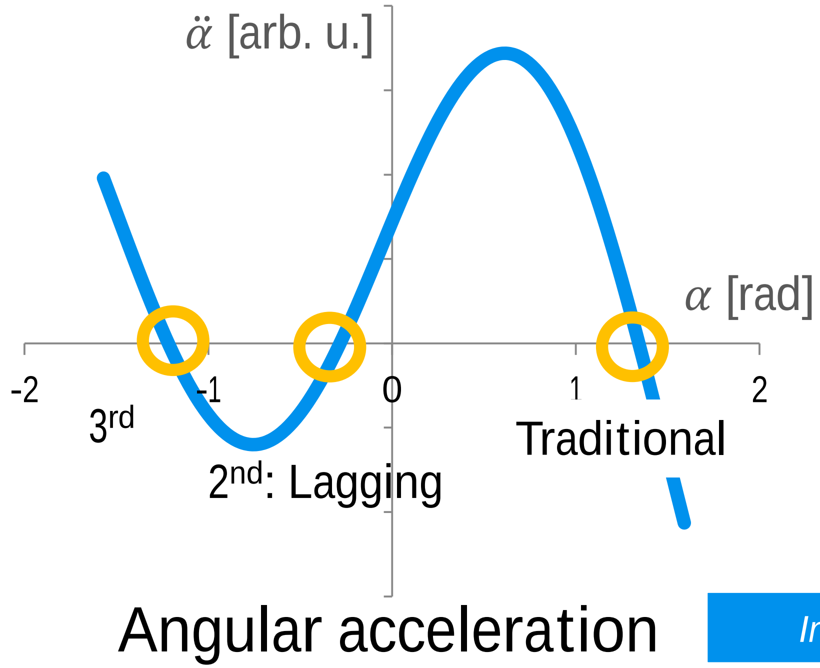
A

Pe

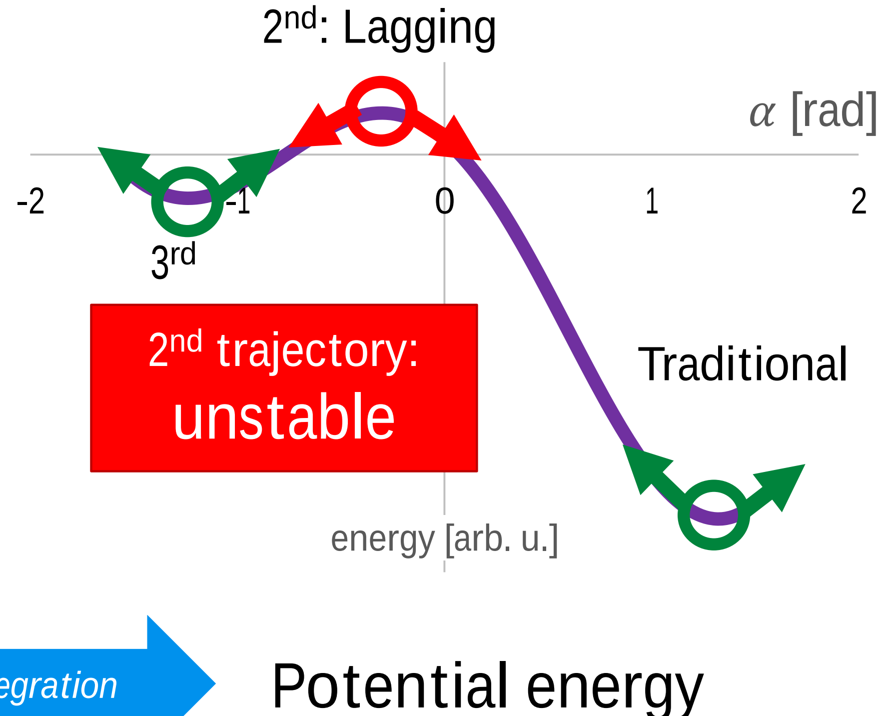
Hold on...

We should also add 2nd dimension

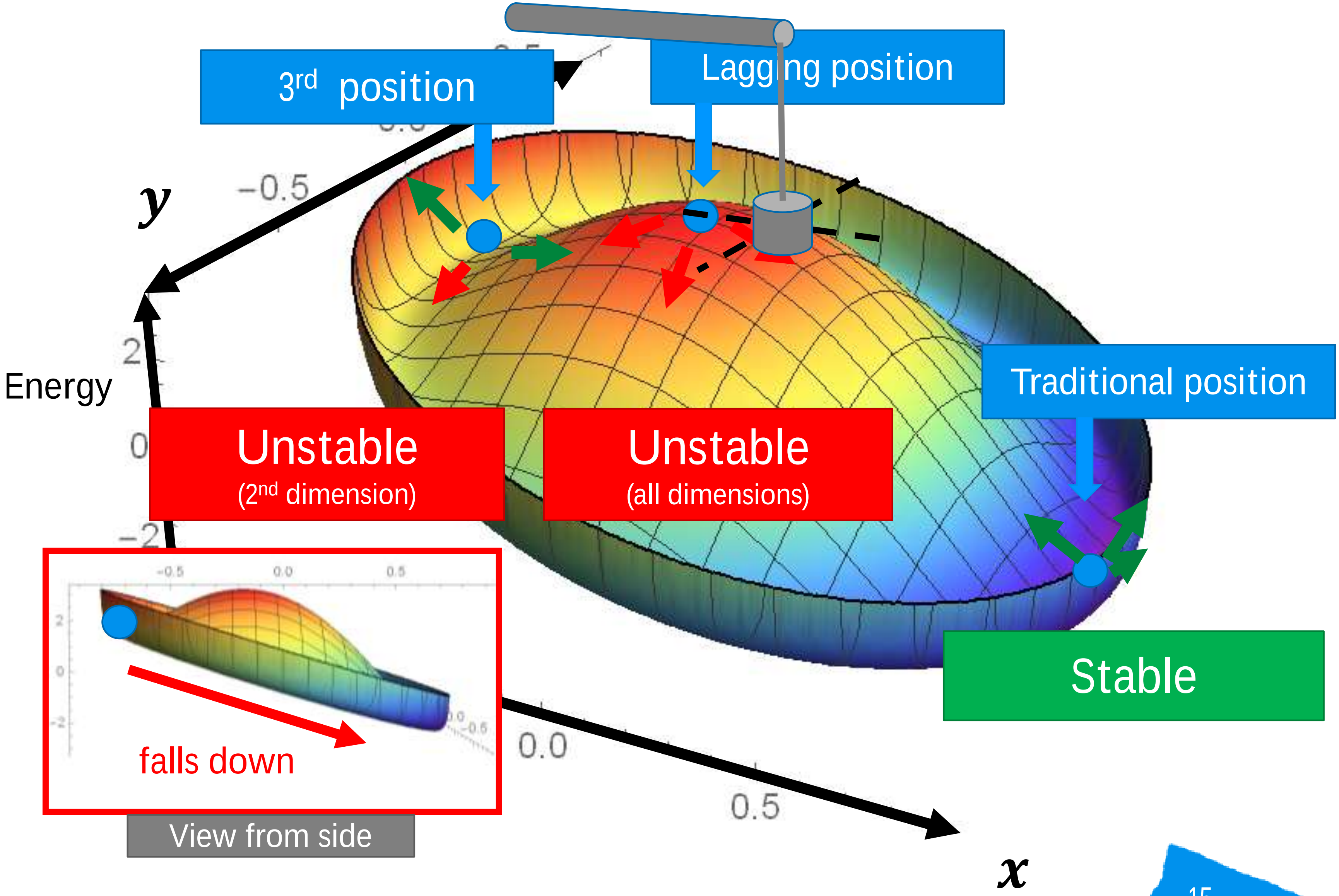
We will plot total effective potential energy



Integration



3D Energy plot in corotating frame



Results

Stability analysis does not correspond with the experiment, where the lagging motion (2^{nd} position) is observed

Something is missing...

We haven't included Coriolis force!

Stabilizing effect of Coriolis force

Coriolis acceleration:

$$\boldsymbol{a}_C = 2\boldsymbol{\omega} \times \boldsymbol{v}$$

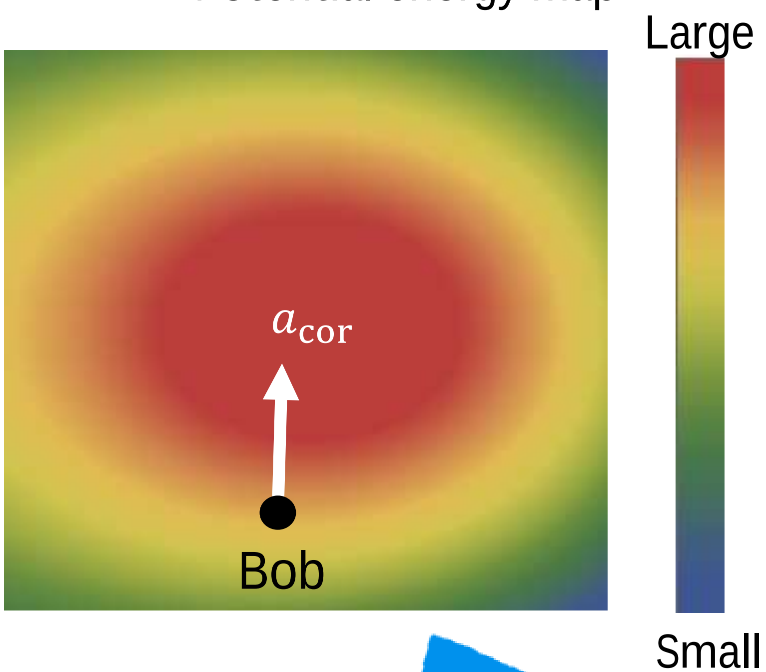
Velocity of the bob in rotating frame

Angular velocity of the rotating frame

Vector product ⇒ perpendicular to both $\boldsymbol{\omega}$ and $\boldsymbol{v}$

Coriolis force could act as a centripetal force
(that has the same direction)

Potential energy map



Stabilizing effect of Coriolis force

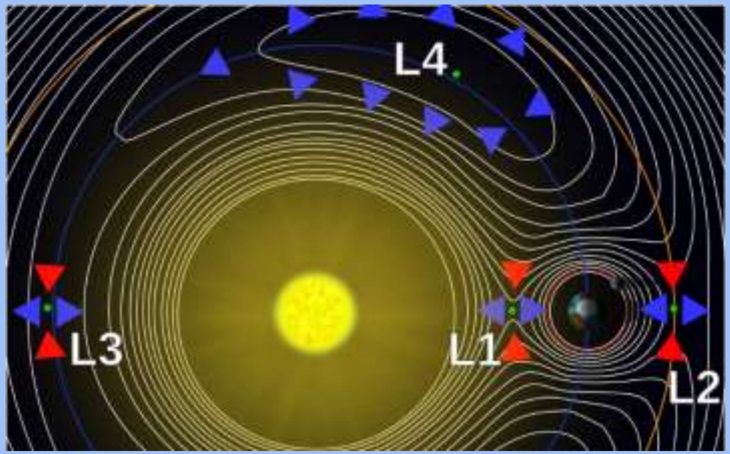
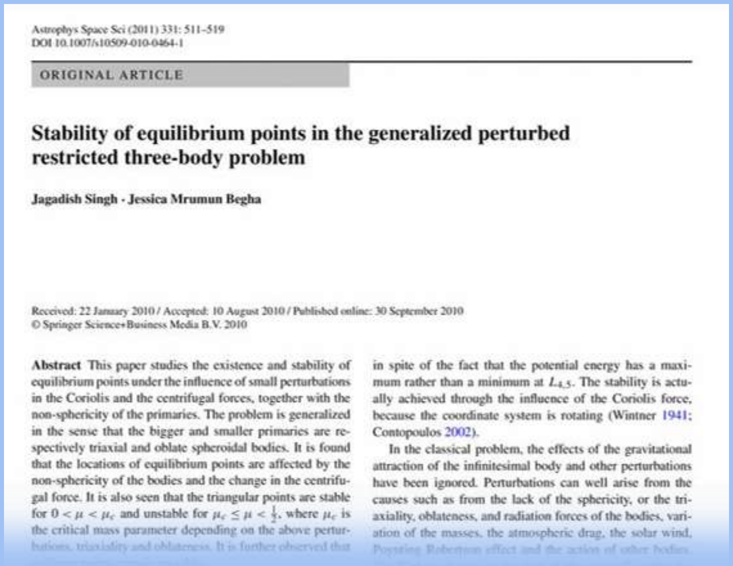


Photo credits:
By Lagrange_points.jpg: created by NASAderivative work: Xander89 - Lagrange_points.jpg, CC BY 3.0,

https://commons.wikimedia.org/w/index.php?curid=7547312

Well-known problem:
stability of L4, L5 Lagrange points
also stabilized by the Coriolis force

Singh, Jagadish, and Jessica Mrumun Begha. "Stability of equilibrium points in the generalized perturbed restricted three-body problem." *Astrophysics and Space Science* 331.2 (2011): 511-519.



Astrophys Space Sci (2011) 331: 511–519
DOI 10.1007/s10509-010-0464-1

ORIGINAL ARTICLE

Stability of equilibrium points in the generalized perturbed restricted three-body problem

Jagadish Singh · Jessica Mrumun Begha

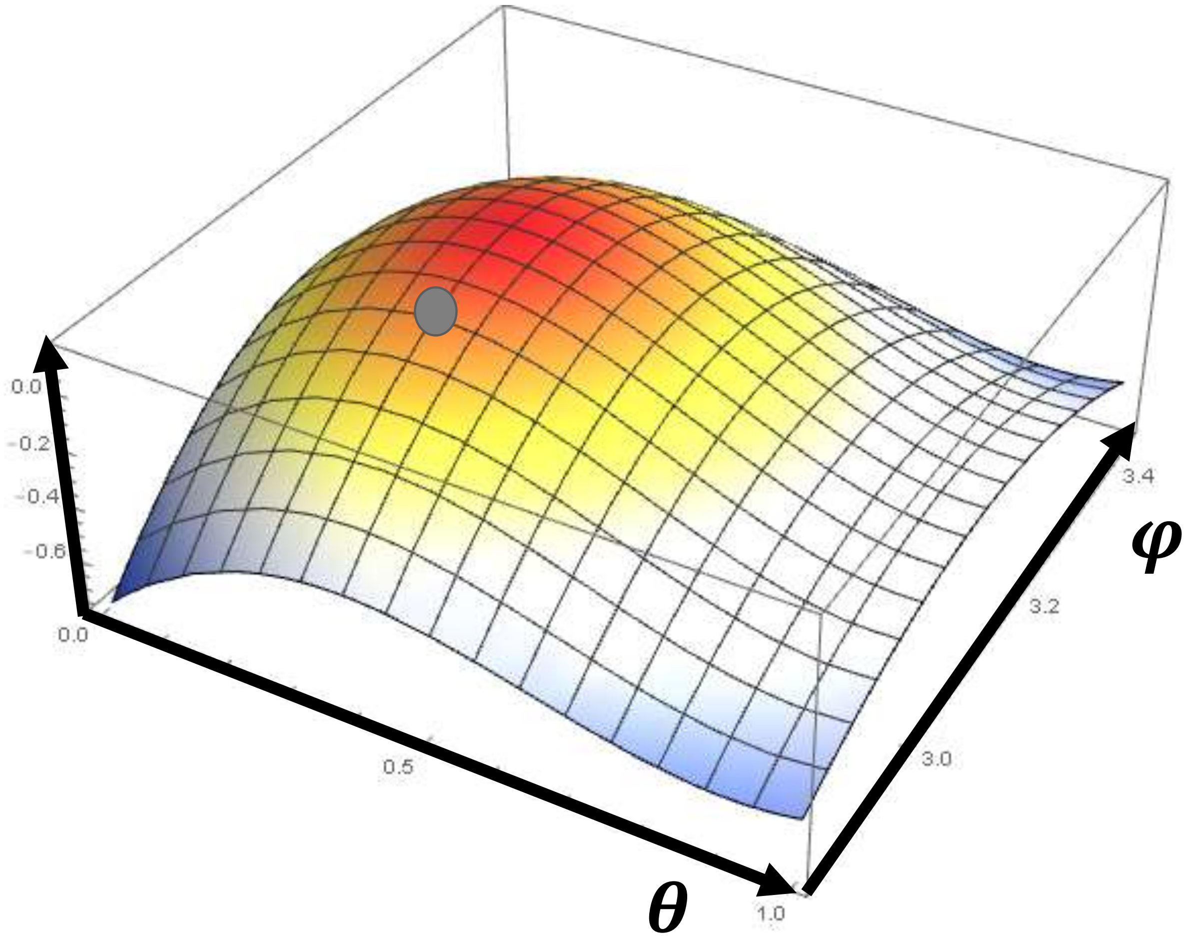
Received: 22 January 2010 / Accepted: 10 August 2010 / Published online: 30 September 2010

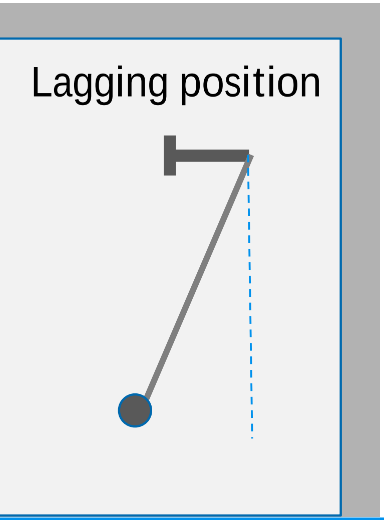
Abstract This paper studies the existence and stability of equilibrium points under the influence of small perturbations in the Coriolis and the centrifugal forces, together with the non-sphericity of the primaries. The problem is generalized in the sense that the bigger and smaller primaries are respectively triaxial and oblate spheroidal bodies. It is found that the locations of equilibrium points are affected by the non-sphericity of the bodies and the change in the centrifugal force. It is also seen that the triangular points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \le \mu < \frac{1}{2}$, where μ_c is the critical mass parameter depending on the above perturbations, triaxiality and oblateness. It is further observed that

in spite of the fact that the potential energy has a maximum rather than a minimum at $L_{4,5}$. The stability is actually achieved through the influence of the Coriolis force, because the coordinate system is rotating (Wintner 1941; Contopoulos 2002).

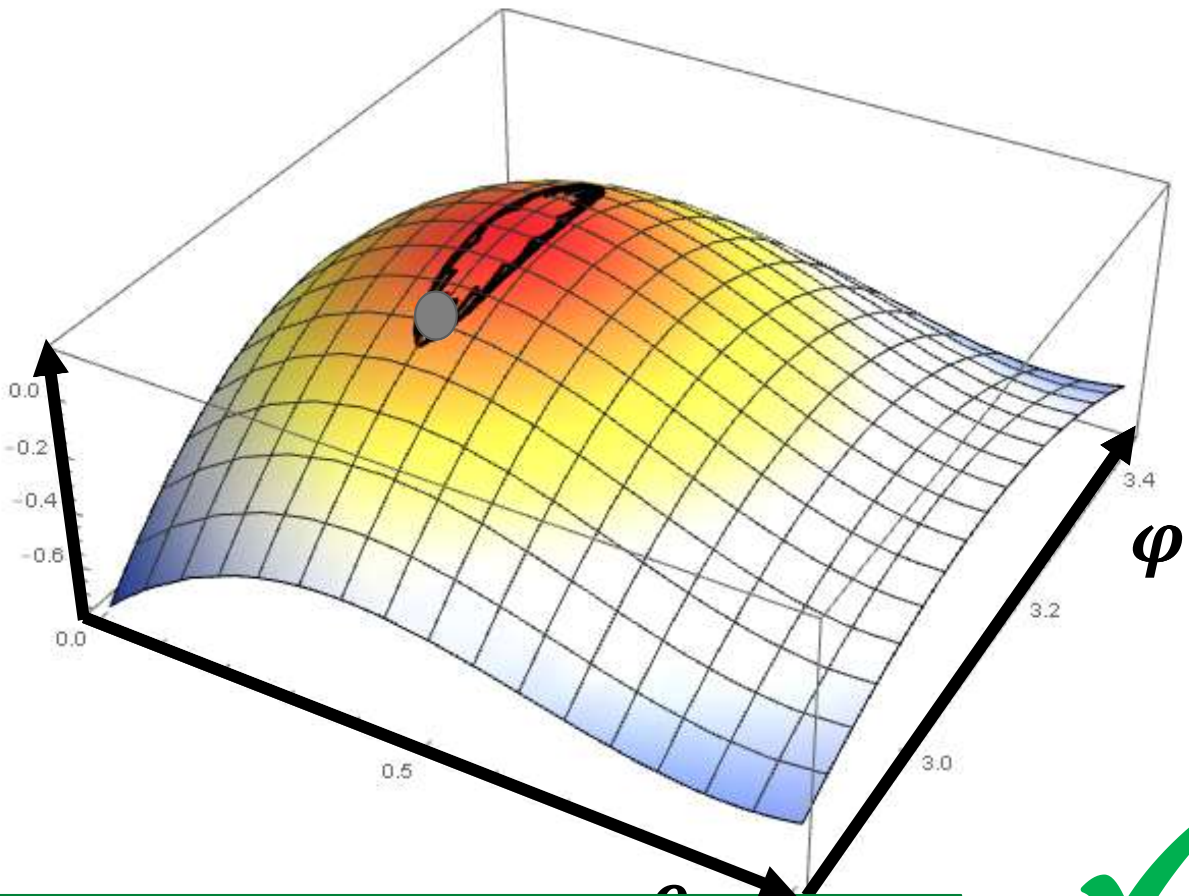
In the classical problem, the effects of the gravitational attraction of the infinitesimal body and other perturbations have been ignored. Perturbations can well arise from the causes such as from the lack of the sphericity, or the triaxiality, oblateness, and radiation forces of the bodies, variation of the masses, the atmospheric drag, the solar wind, Poynting Robertson effect and the action of other bodies.

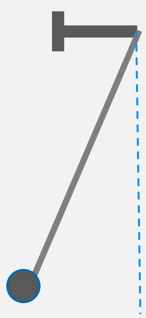
How does Coriolis stabilize?





How does Coriolis stabilize?

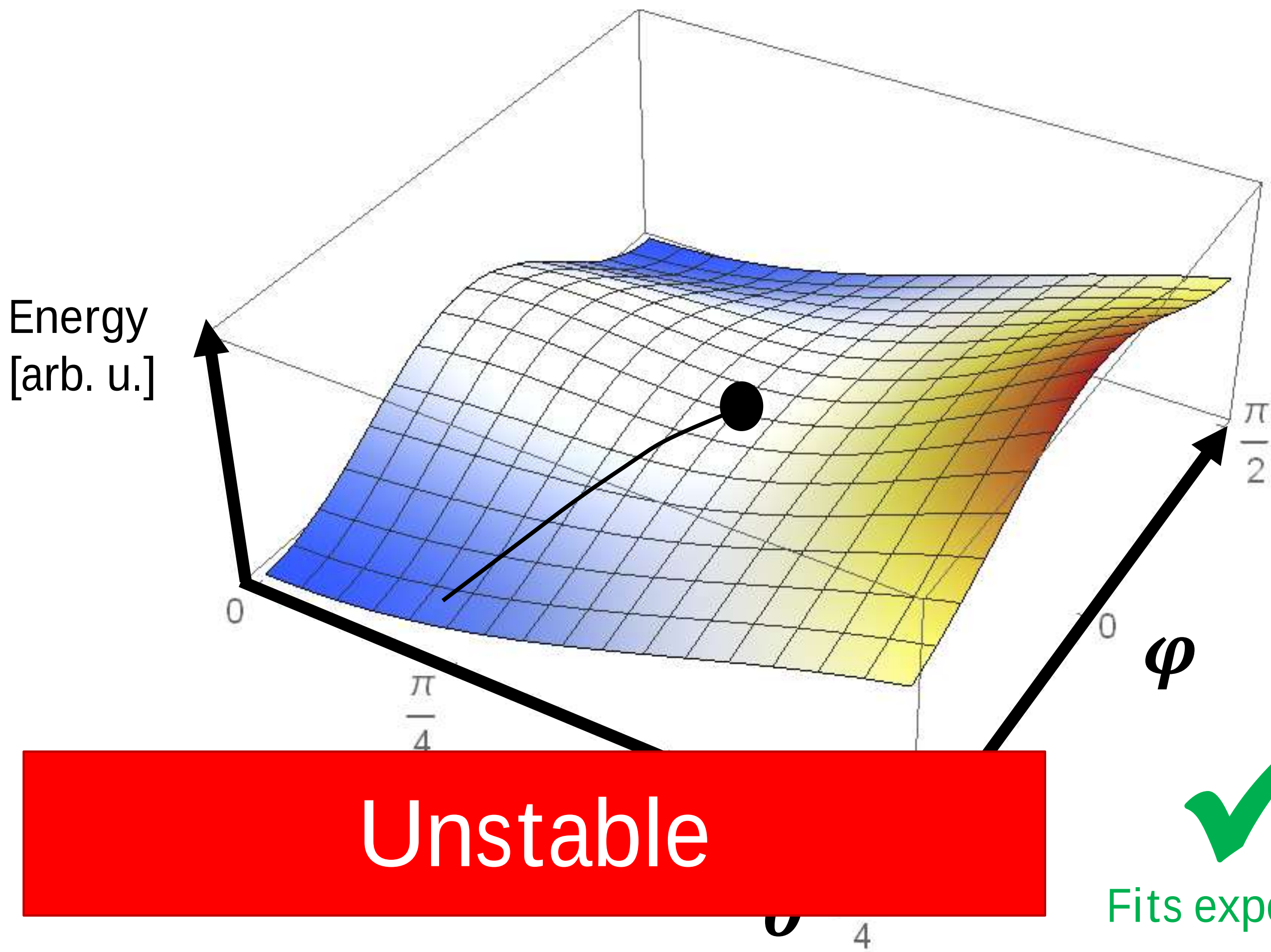




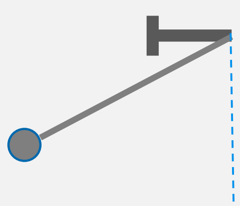
Stable

Fits experiment

Could Coriolis force stabilize 3rd position?



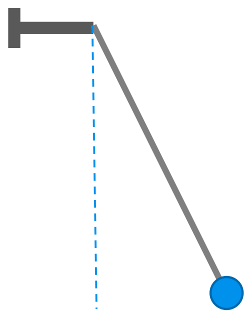
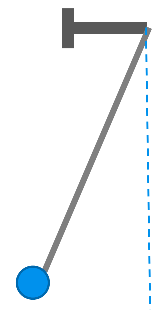
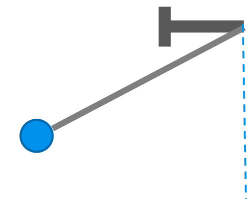
3rd position



Unstable

Fits experiment

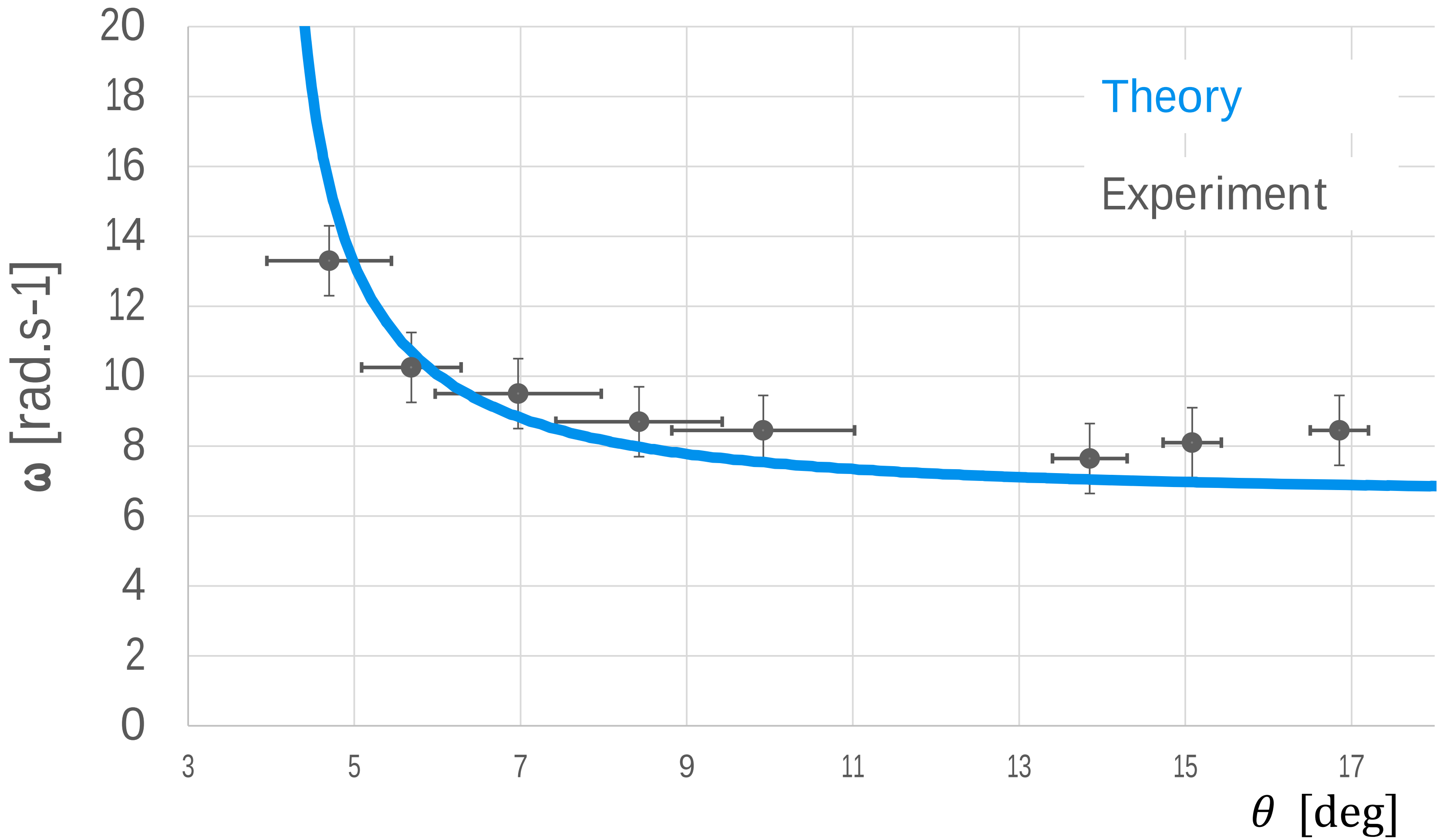
Overview: predictions and experiment

	1st trajectory: "traditional"	2nd trajectory: "lagging"	3rd trajectory
Position:			
Theoretical predictions:	Stable	Stable	Unstable
Experiment:	✓ Observed	✓ Observed	✓ Not observed

Qualitative predictions → Quantitative predictions

✓ Done

Lagging angle: theory – experiment



Task

A pendulum consists of a strong thread and a bob.

When the pivot of the pendulum starts moving along a horizontal circumference, the bob starts tracing a circle which can have a smaller radius, under certain conditions.

Investigate the motion and stable trajectories of the bob.

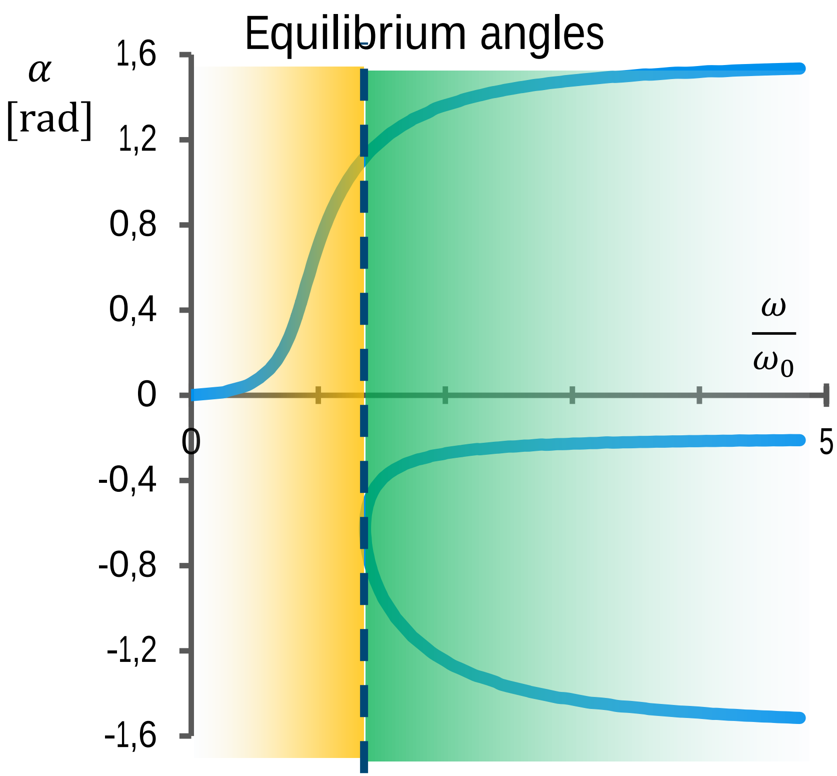
✓ tracing a circle ✓ smaller radius

Under certain conditions...

CONDITIONS OF LAGGING MOTION

1. Parameters of pendulum: $R/L, \omega/\omega_0$
2. Initial conditions

Parameters: ω/ω_0 by R/L

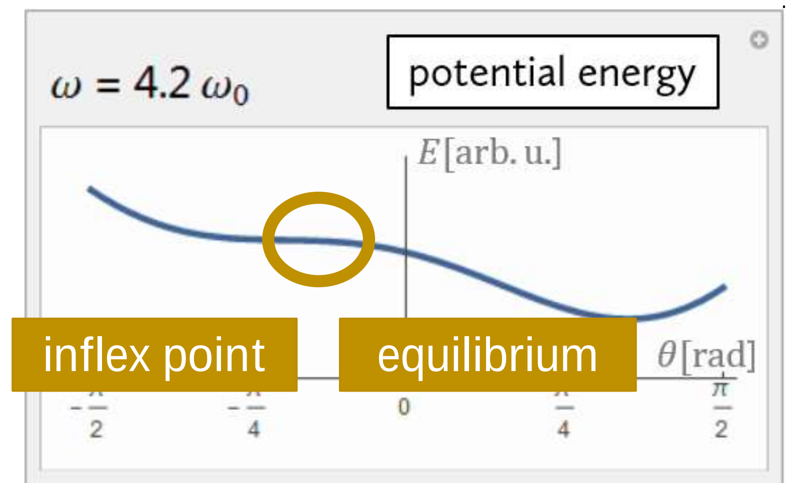


Only 1 position

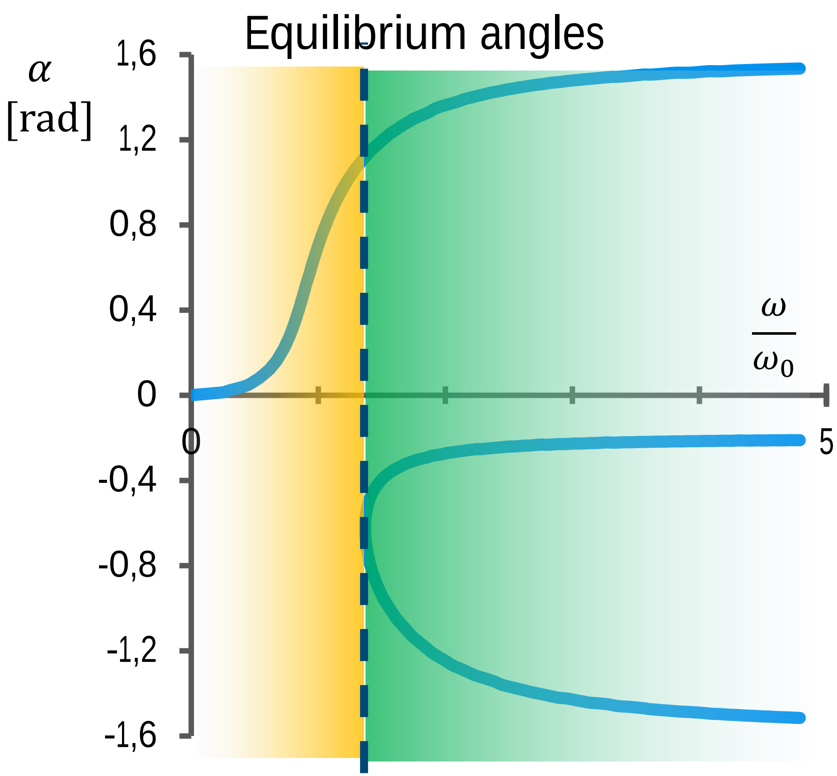
Lagging position exists

Critical angular velocity ($\boldsymbol{\omega_c}$)

We can calculate critical angular velocity from theory



Parameters: ω/ω_0 by R/L

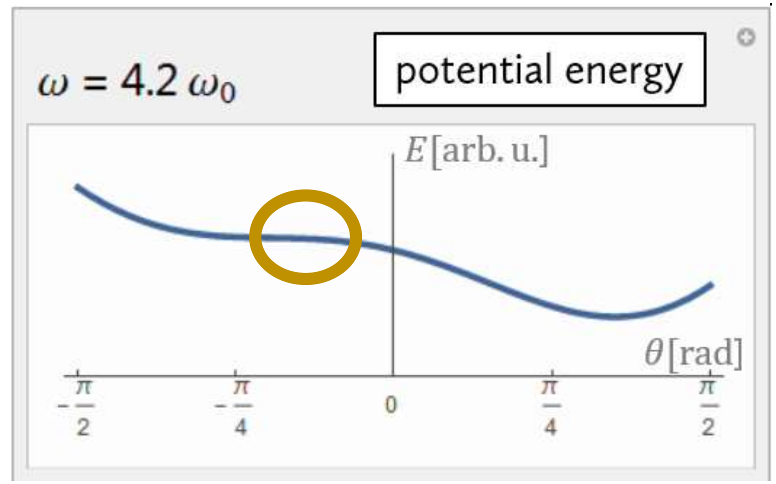


Only 1 position

Lagging position exists

Critical angular velocity ($\boldsymbol{\omega_c}$)

We can calculate critical angular velocity from theory



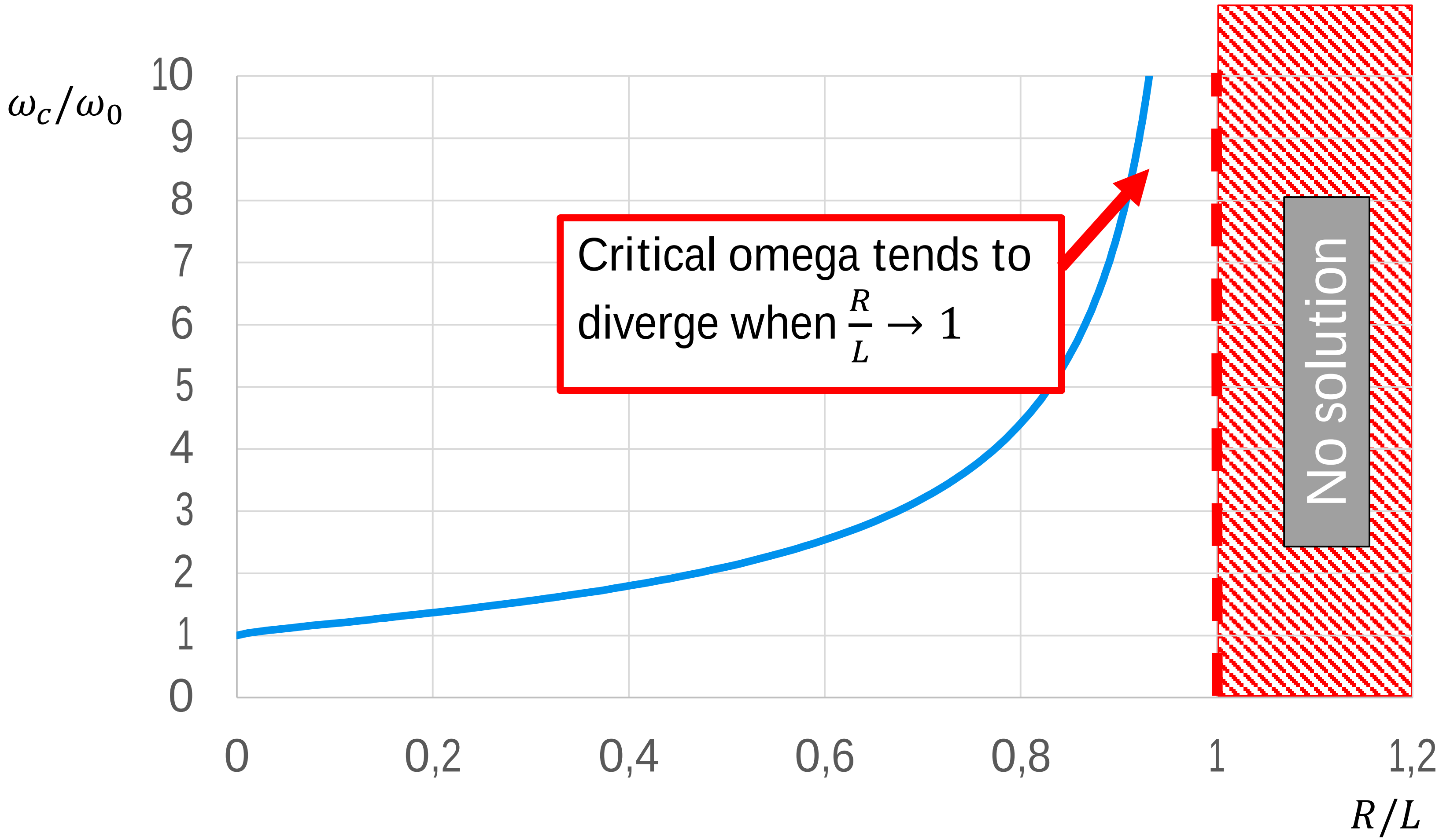
conditions:

equilibrium $\left.\frac{\partial E}{\partial \alpha}\right|_{\omega_c} = 0$

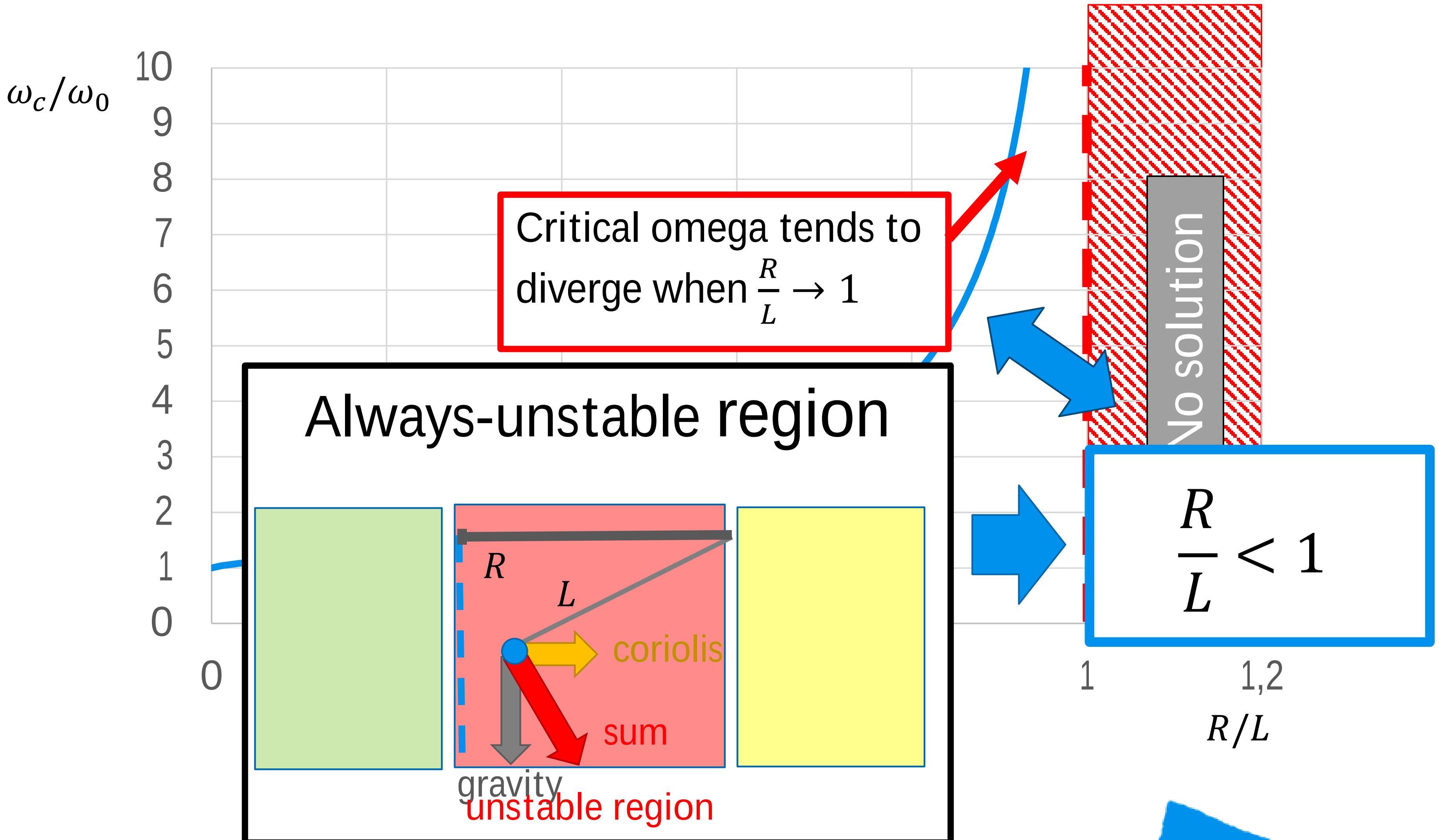
inflex point $\left.\frac{\partial^2 E}{\partial \alpha^2}\right|_{\omega_c} = 0$

Can be solved only numerically

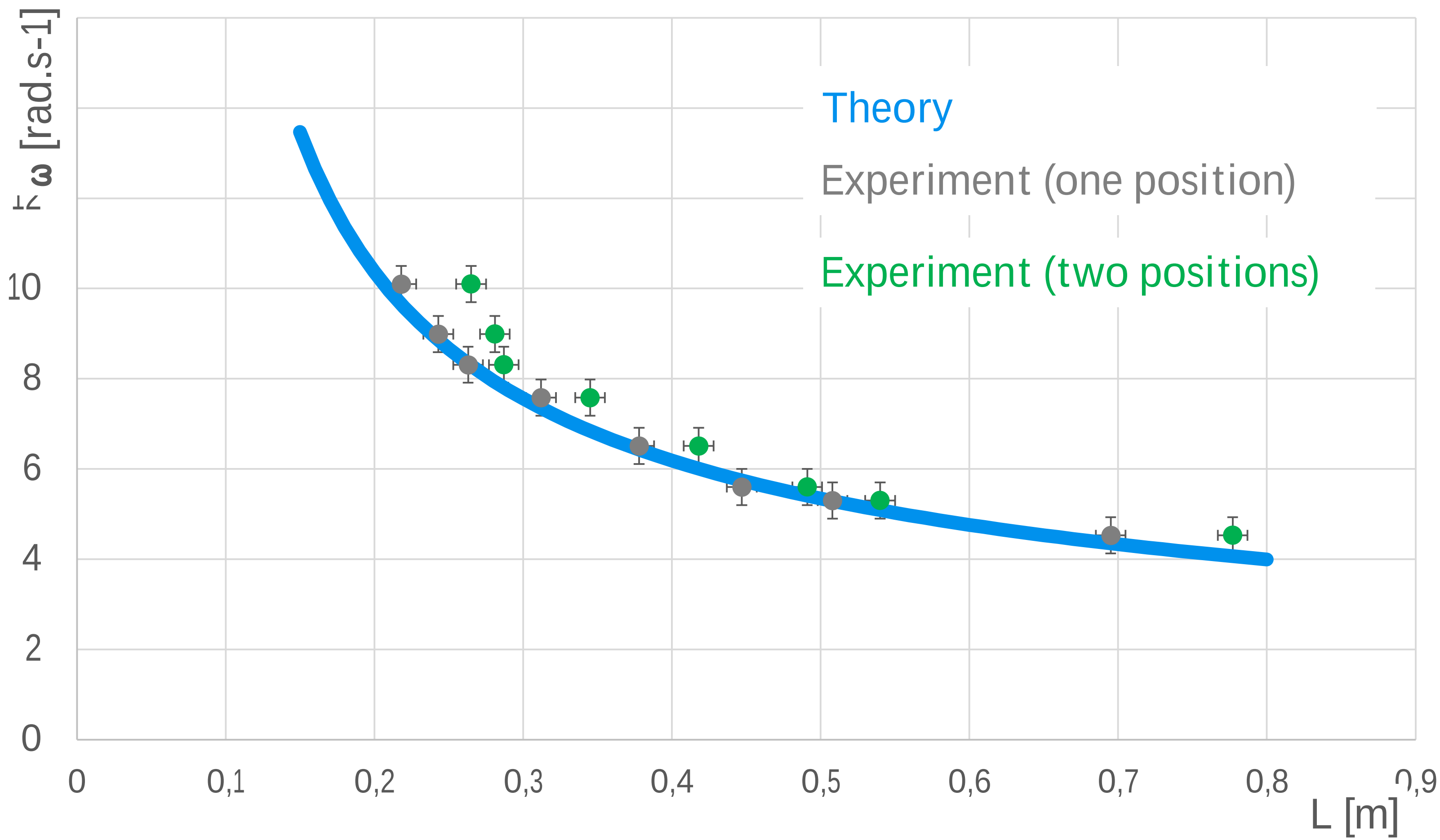
Critical omega – theoretical prediction



Critical omega – theoretical prediction



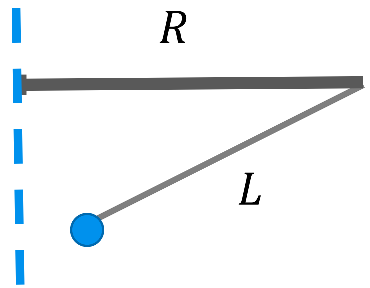
Critical angular velocity: theory-experiment



Conditions for parameters

1. Ratio of lengths:

$$\frac{R}{L} < 1$$



2. Angular velocity:

$$\omega > \omega_c$$

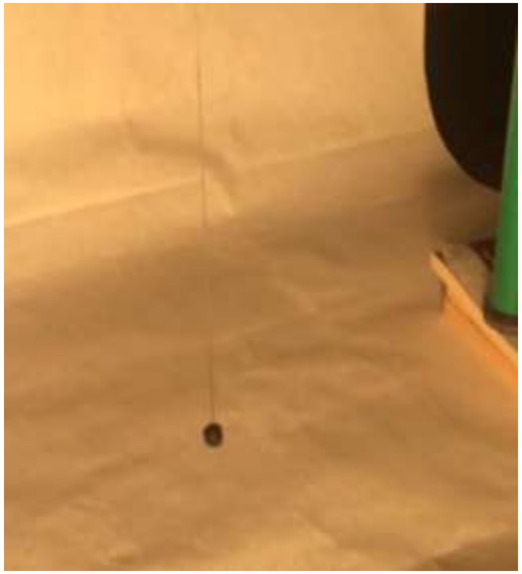
Where ω_c is a function of $\frac{R}{L}$ and $\omega_0 = \sqrt{\frac{g}{L}}$

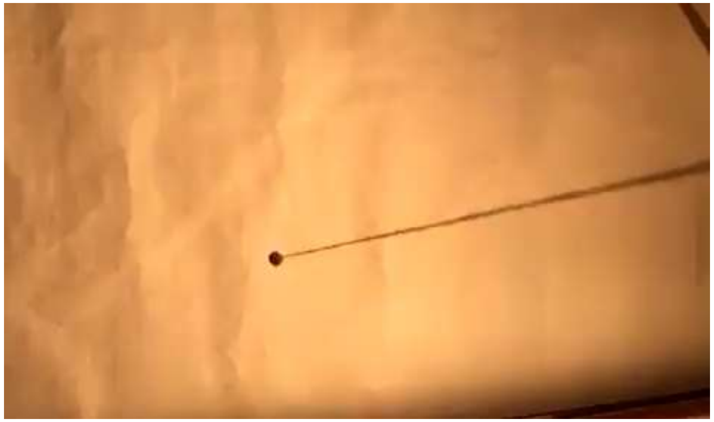
Conditions: parameters necessary for lagging motion

Initial conditions: time evolution

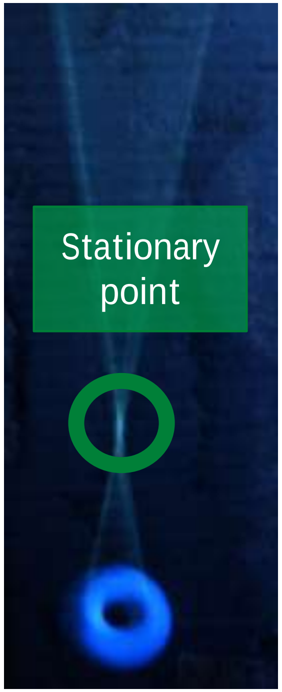
In the beginning the motion is jerky (trembling)

Later, bob stabilizes to lagging motion





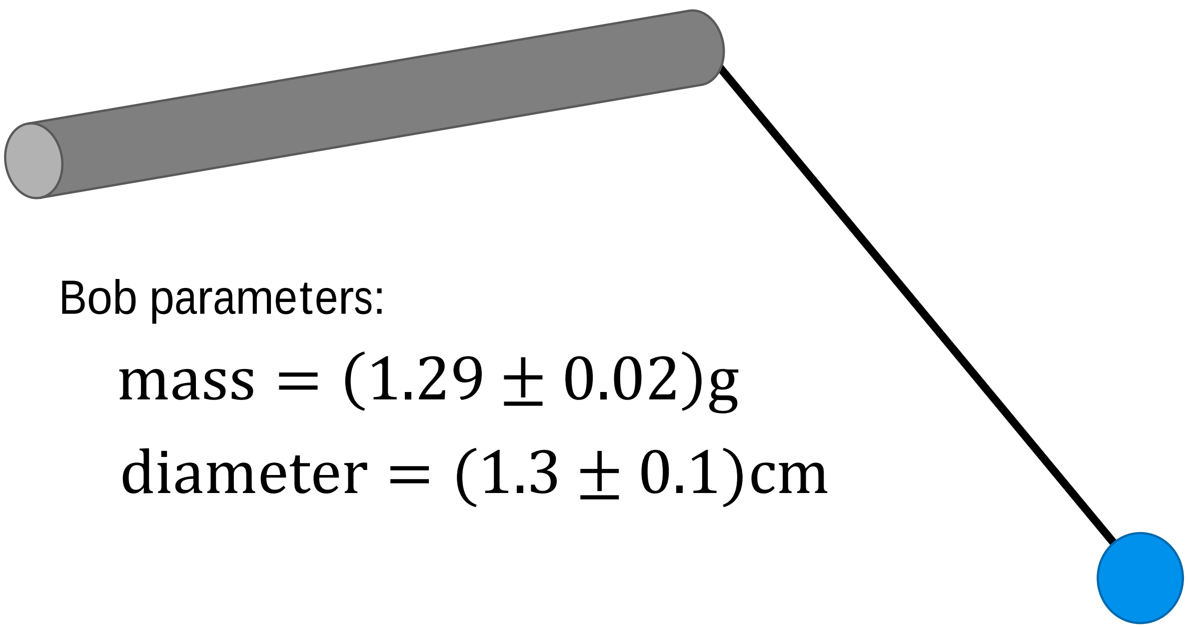
Time-lapse video: trembling → lagging



Lagging motion: Long-exposure photograph

Motion is being stabilized by air drag

Effect of air drag



Bob parameters:

$$\text{mass} = (1.29 \pm 0.02)\text{g}$$

$$\text{diameter} = (1.3 \pm 0.1)\text{cm}$$

$$a_{\text{drag}} = \boxed{\frac{\frac{1}{2}CS\rho}{m}} v^2$$

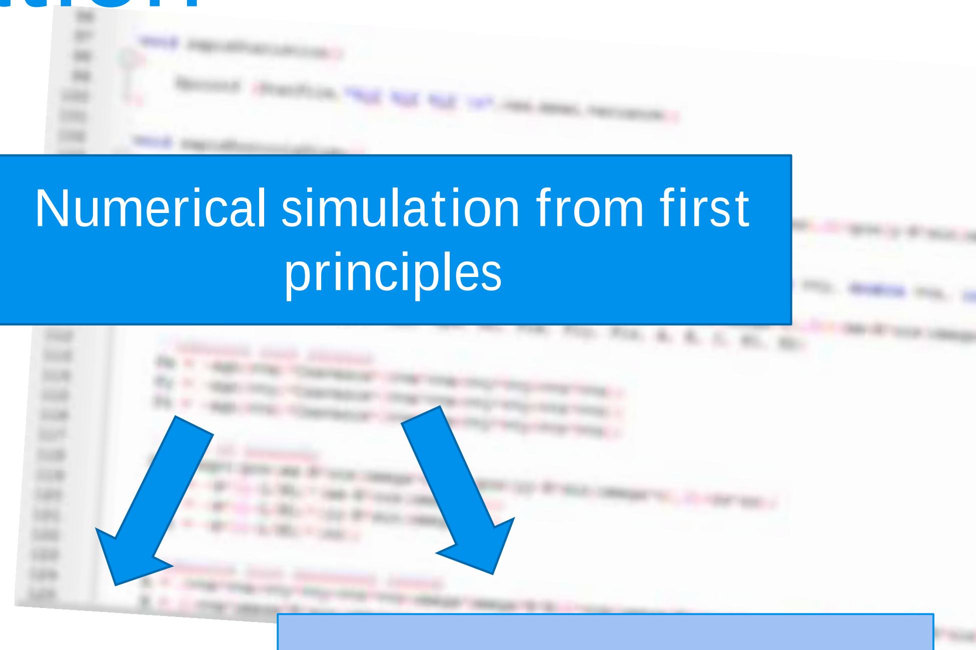
$$\boldsymbol{c} = \frac{\frac{1}{2}CS\rho}{m}$$

"Effective drag coefficient"

Object	Effective drag coefficient
Bob	$\approx 0{,}03\ \text{m}^{-1}$
Thread	$\approx 0{,}10\ \text{m}^{-2} \cdot L$ $(\text{L} \approx 0.5\text{m} - 1\ \text{m})$

Thread has also a great affect on damping

Numerical simulation



We are unable to solve system analytically

Numerical simulation from first principles

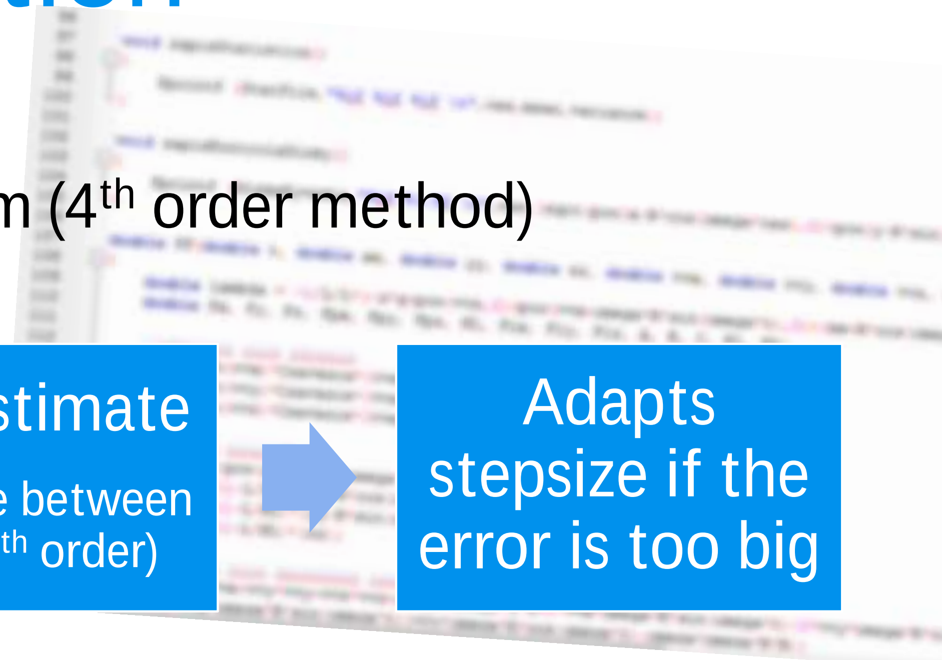
Inertial frame of reference used

Constraint modelled using Lagrange multipliers

Air drag of the thread computed
(may have different direction than that of the bob)

(Spherical coordinates omitted due instabilities near poles)

Numerical simulation



Runge-Kutta-Fehlberg algorithm (4th order method)

Computes both 4th order and 5th order → Error estimate (difference between 4th and 5th order) → Adapts stepsize if the error is too big

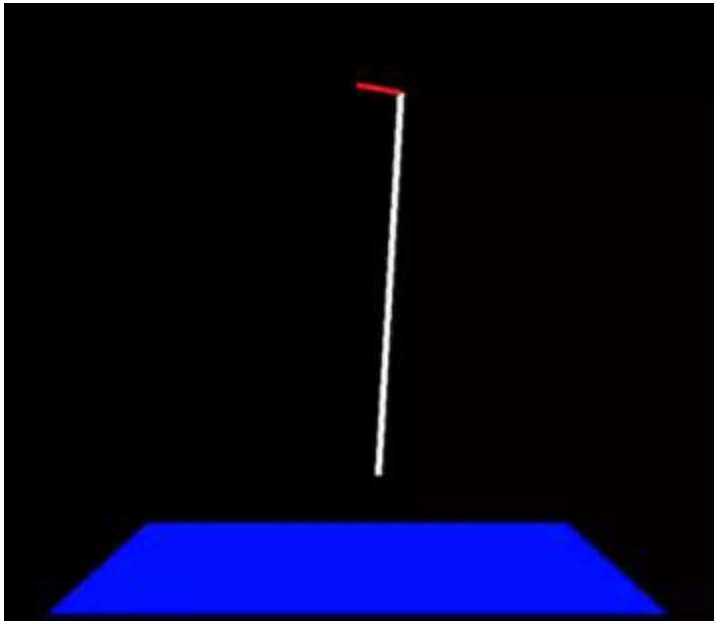
Despite these upgrades:

- Numerical instability (violation of constraint)

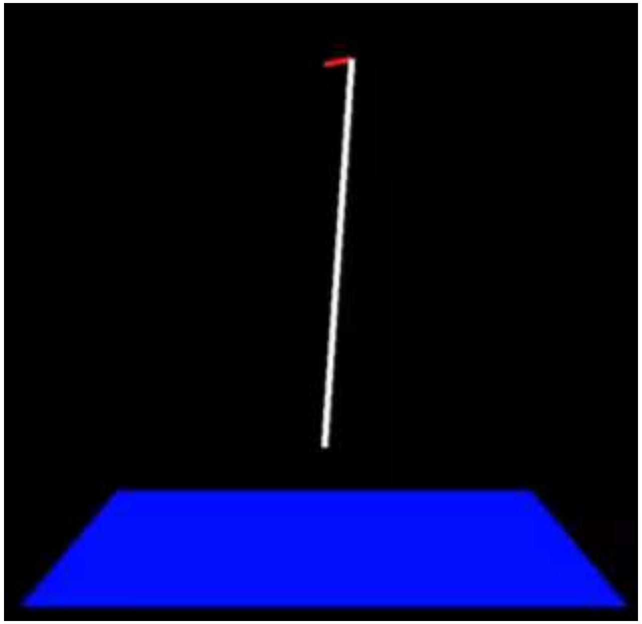
➡ Thread exchanged with very hard spring

The lenght of thread stays constant in simulation & spring does not influence character of motion

Example output from simulation



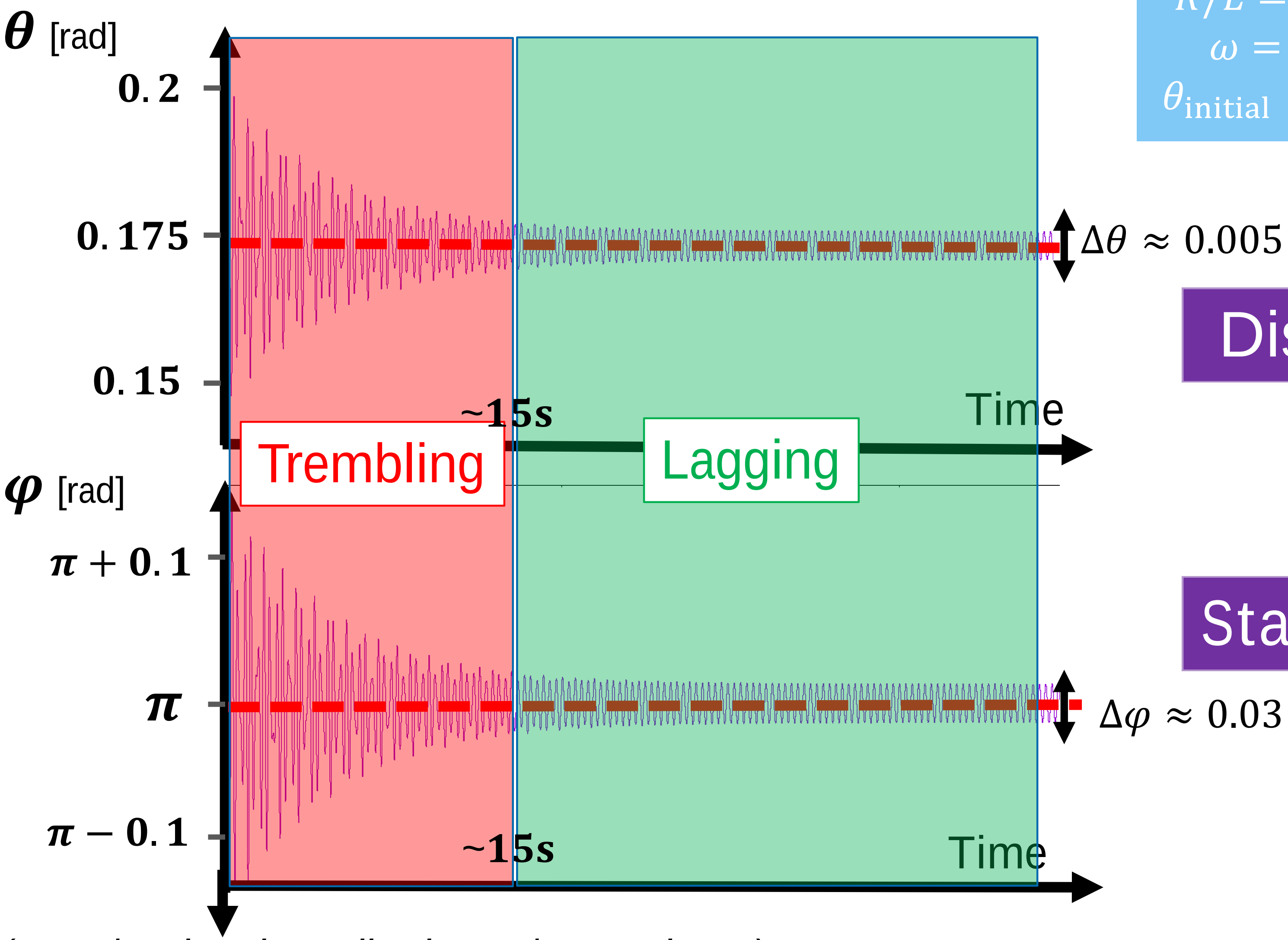
Trembling
(initial motion)



Lagging motion
(relaxed motion)

Air drag: stabilization

Parameters:
$R/L \doteq 0.114$
$\omega = 4\omega_0$
$\theta_{\text{initial}} = 0.15$



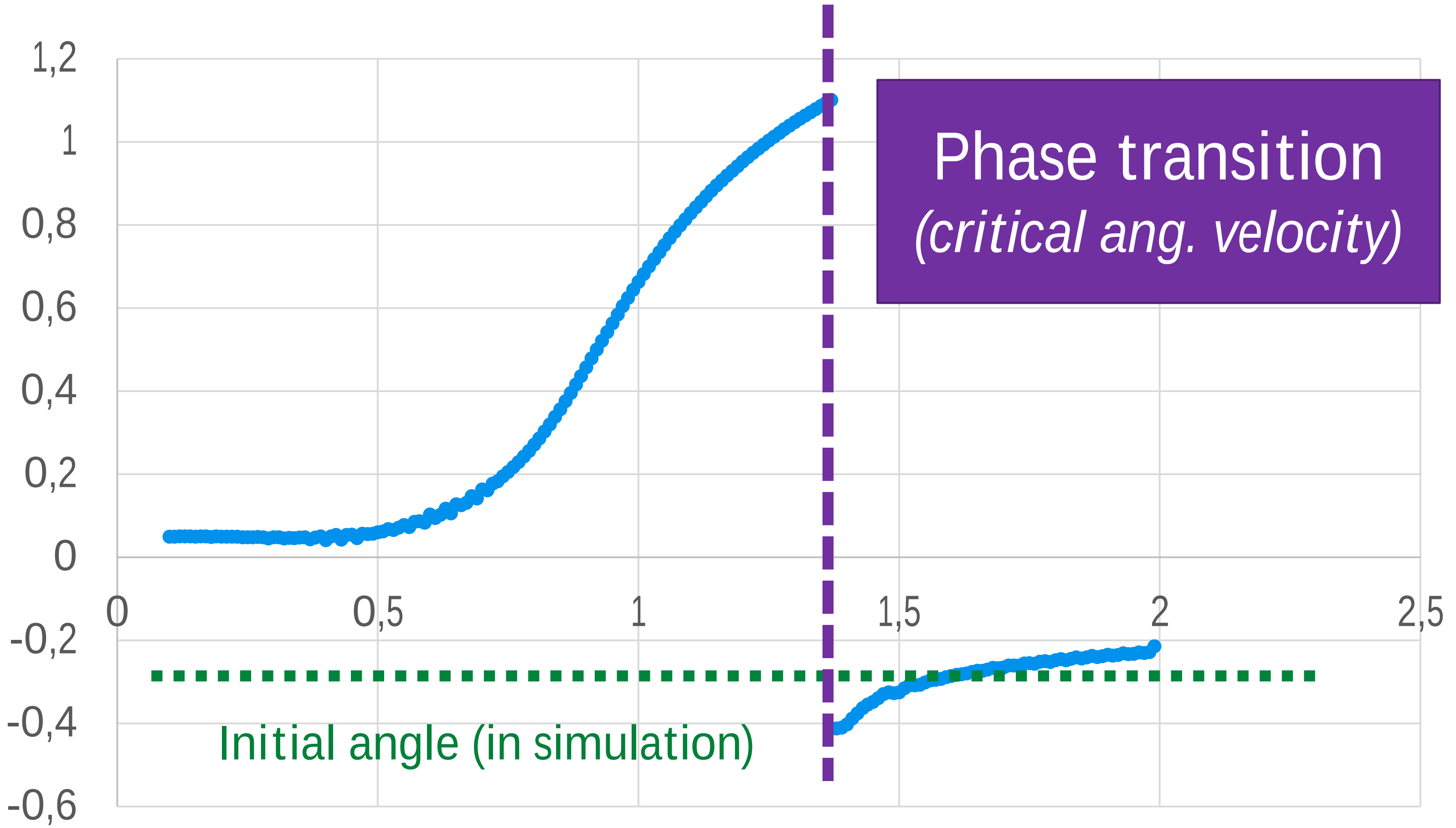
Dissipation

↓

Stabilization

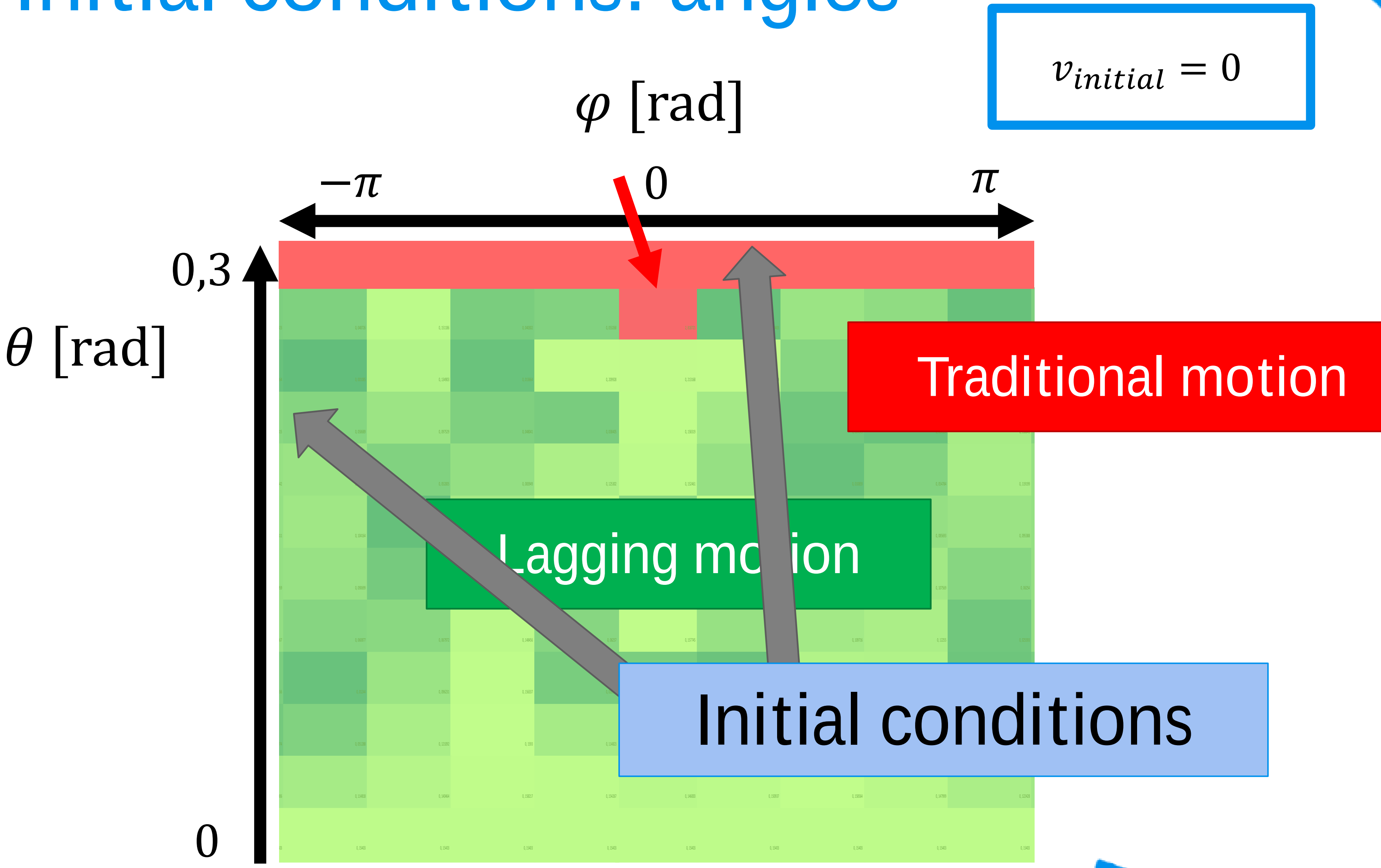
(Boundary is only qualitative and approximate)

Simulation: equilibrated angles θ



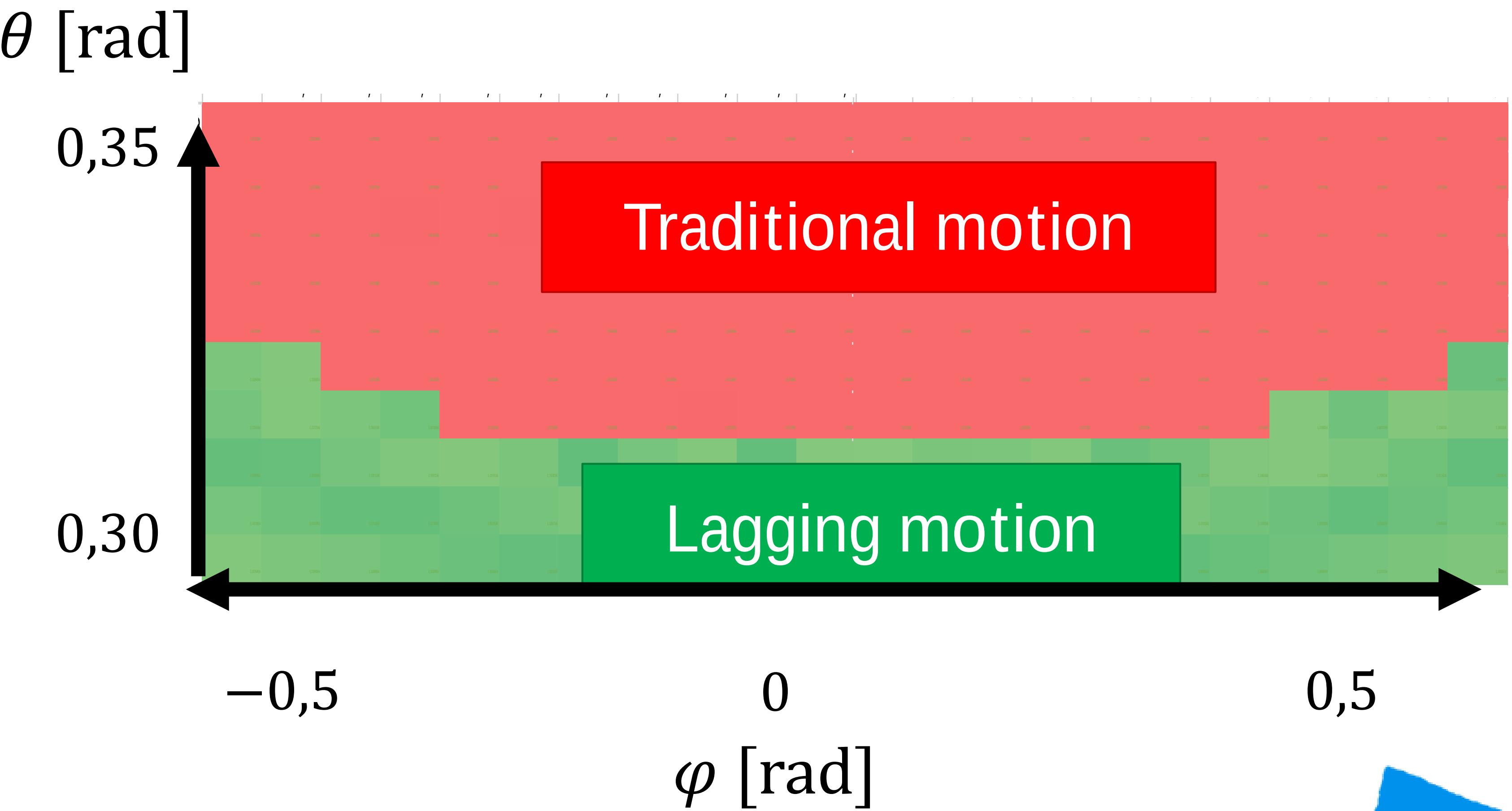
Changes in θ created by drag are negligible

Initial conditions: angles



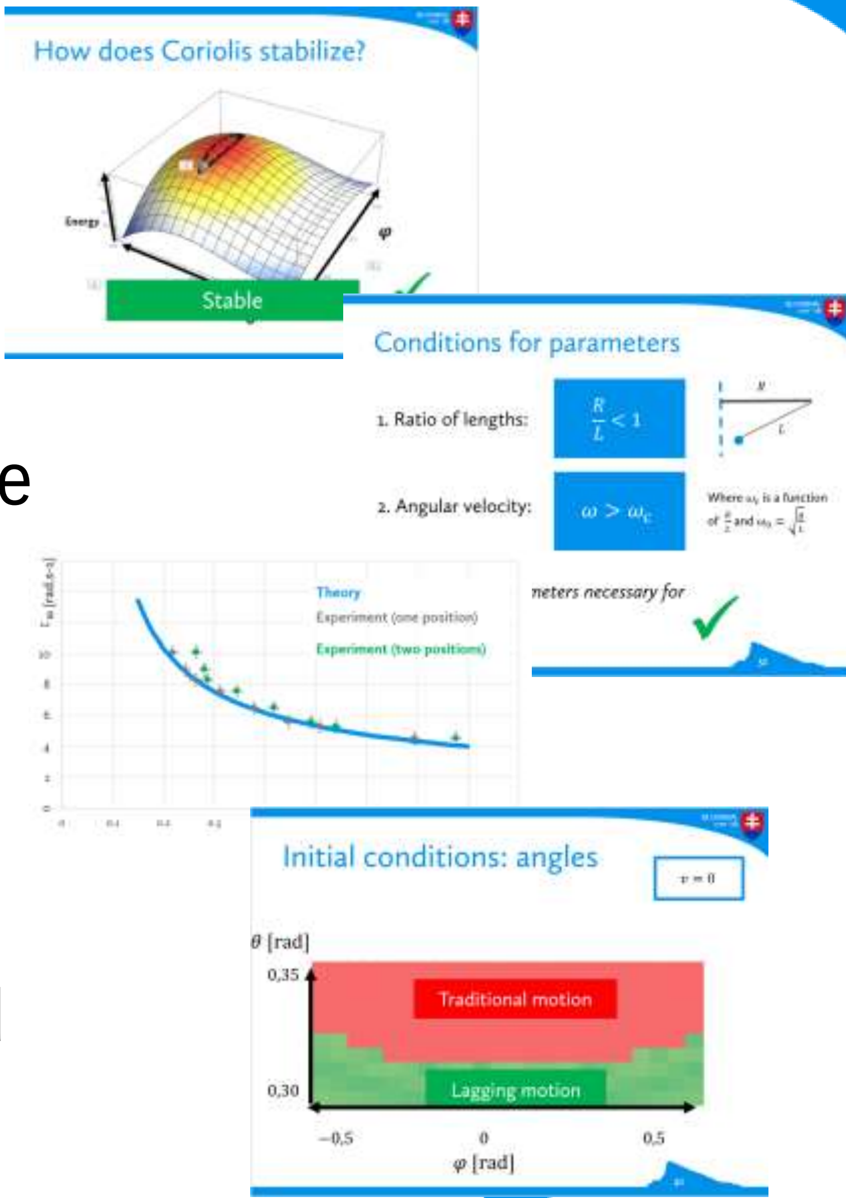
Initial conditions: angles

$v = 0$



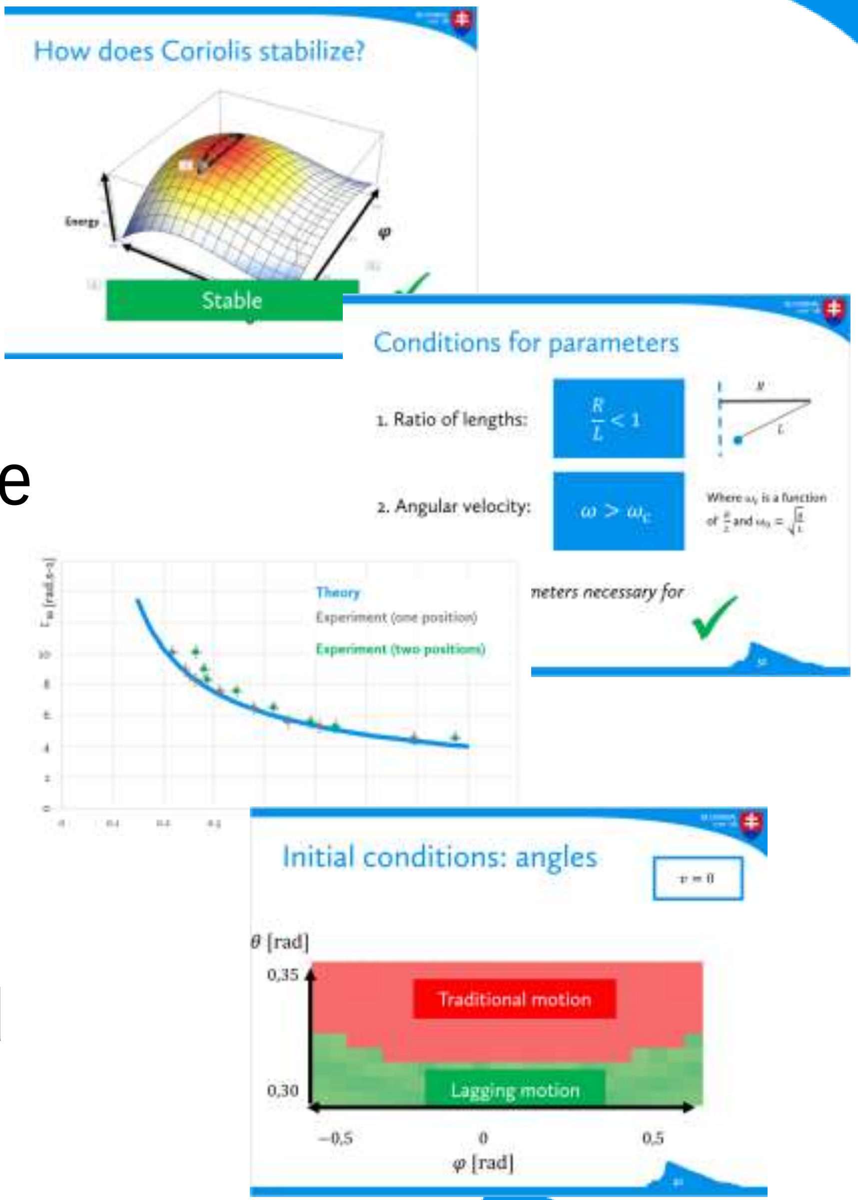
Conclusion

1. Lagging motion at a potential maximum
   – Stabilized by Coriolis force
2. Conditions for lagging motion
3. 3D equations of motion
   – Phase diagrams predicted



Thank you for your attention!

1. Lagging motion at a potential maximum
   – Stabilized by Coriolis force
2. Conditions for lagging motion
3. 3D equations of motion
   – Phase diagrams predicted



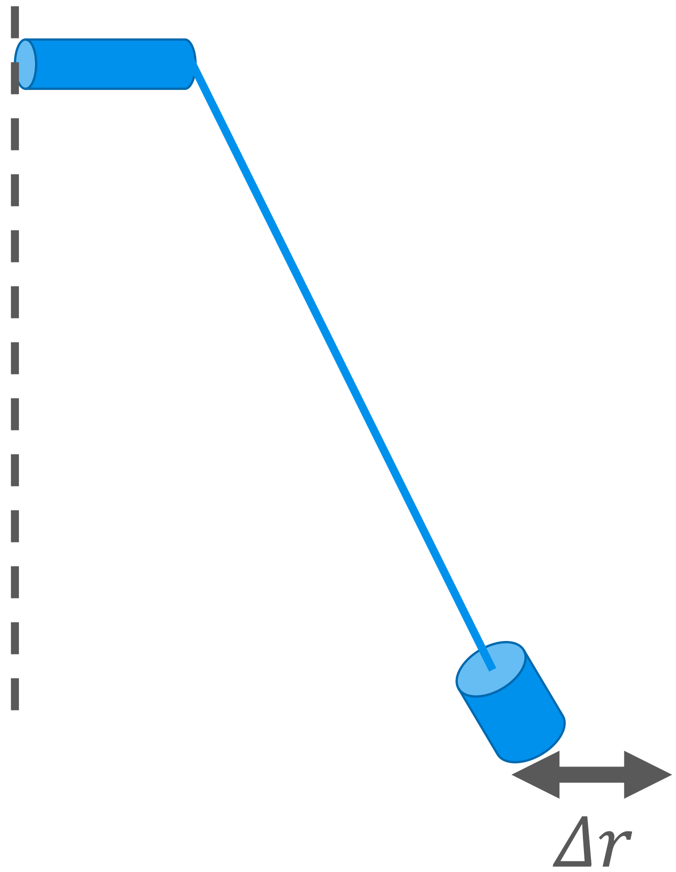
02

Lagging Pendulum

Martin Gazo

APPENDICES

What is Coriolis force?

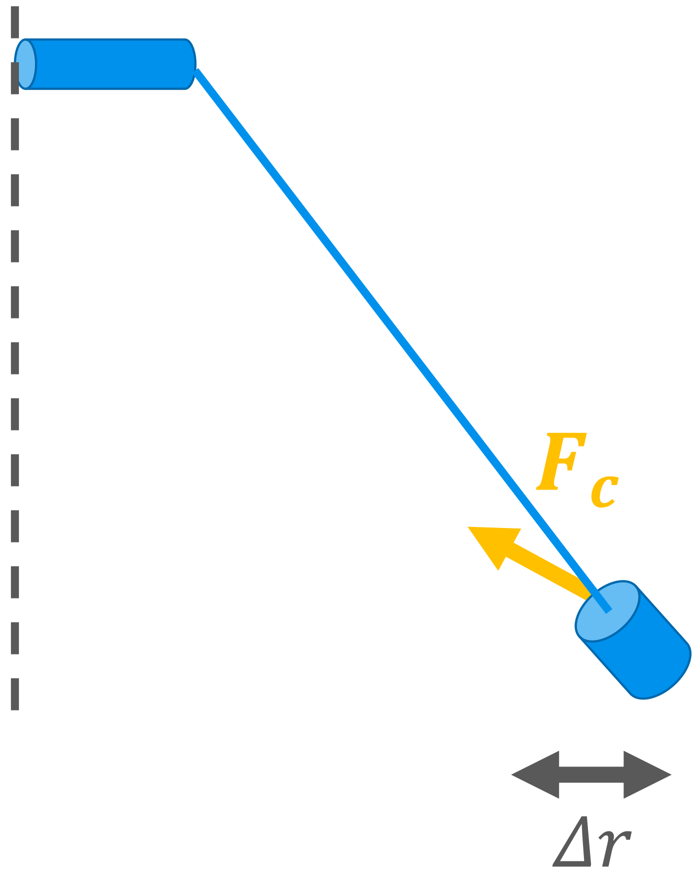


Bob moves away from the center

Coriolis acceleration:

$$\boldsymbol{a}_C = 2\boldsymbol{\omega} \times \boldsymbol{v}$$

What is Coriolis force?



Bob moves away from the center

Velocity change
$\Delta v = \omega \Delta r$
(in tangential direction)

Fictitious force acting sideways

Coriolis acceleration:

$$\boldsymbol{a}_C = 2\boldsymbol{\omega} \times \boldsymbol{v}$$

Coriolis on parabolic potential

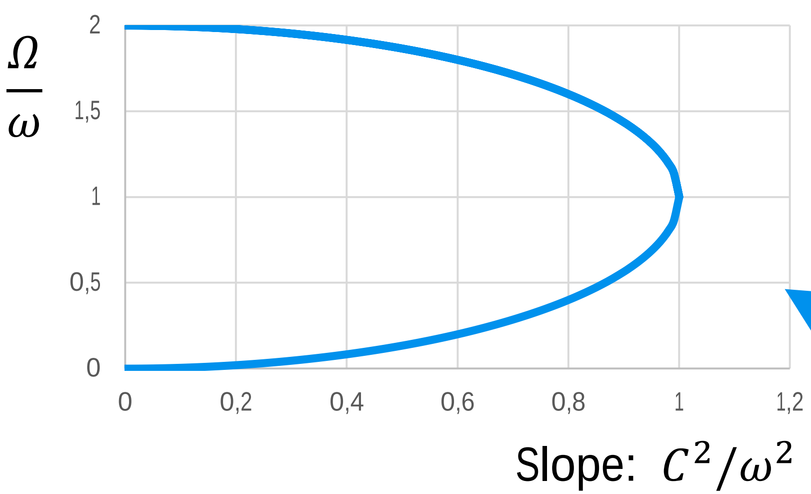
Circular trajectories around equilibrium

$$\Omega^2 r = -C^2\, r + 2\,\omega\,(\Omega r)$$

Centripetal

Centrifugal + Gravity
(2nd order Taylor expansion of potential)

Coriolis



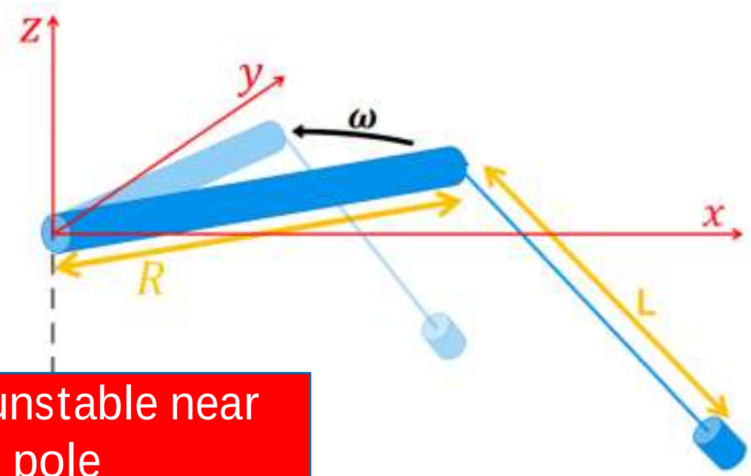
Ω – angular frequency of oscillation around equilibrium

Solution:

$$\Omega = \omega \pm \sqrt{\omega^2 - C^2}$$

No solution for steep potential

Simulation: Equations of motion



Lagrange approach

Spherical coordinates θ, φ ✗

$$\ddot{\varphi} \sim \frac{1}{\sin\theta}$$ → Numerically unstable near south pole

Cartesian coordinates x, y, z

$$m\ddot{x} = 0 - 2\lambda(x - R\cos\omega t) + F_{Vx}$$
$$m\ddot{y} = 0 - 2\lambda(y - R\sin\omega t) + F_{Vy}$$
$$m\ddot{z} = -gm - 2\lambda z + F_{Vz}$$

Non-conservative forces (not included in potential energy; e.g. air drag)

Lagrange multiplicator

$$\lambda = -\frac{m}{2L^2}\left(gz - (\dot{x} + \omega R\sin\omega t)^2 - (\dot{y} - \omega R\cos\omega t)^2 - \dot{z}^2 \right.$$
$$-\omega^2 R(\cos\omega t(x - R\cos\omega t) + \sin\omega t(y - R\sin\omega t))$$
$$\left. + F_{Vx}(x - R\cos\omega t) + F_{Vy}(y - R\sin\omega t) + F_{Vz} z \right)$$

Fundamental equations

Definition of Lagrangian

Kinetic and (total) potential energy

$$L = E_k - E_p$$

For Lagrangian applies:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = 0$$

Arbitrary coordinate

$$\dot{s} = \frac{ds}{dt}$$

If there are other forces (not yet included) then:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = \sum \vec{F}_i$$

Sum of all (generalized) forces which have not been included yet in potential energies

Fundamental equations

Definition of Lagrangian

Kinetic and (total) potential energy

$$L = E_k - E_p - \lambda\phi$$

Lagrange multiplier

Constraint equation:

$$\phi(x, y, z) = \underbrace{(x - R cos\omega t)^2 + (y - R sin\omega t)^2 + z^2}_{} - l^2 = 0$$

Determines area where there the thread could be located (because of configuration in which it is)

Fundamental equations

Kinetic energy

Constraint equation

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz - \lambda[(x - Rcos\omega t)^2 + (y - Rsin\omega t)^2 + z^2 - l^2]$$

(total) potential energy

Lagrange multiplier

Counting derivation

$$\frac{\partial L}{\partial \dot{x}} = \frac{1}{2}m2\dot{x}$$

continuing

$$\ddot{x} = 0 - \frac{\lambda}{m}2(x - Rcos\omega t)$$

Other two coordinates will be derived analogically

$$\ddot{y} = 0 - \frac{\lambda}{m}2(y - Rsin\omega t)$$

$$\ddot{z} = -g - \frac{\lambda}{m}2z$$

Fundamental equations

$$m\ddot{x} = 0 - 2\lambda(x - R\cos\omega t) + F_{px}$$

$$m\ddot{y} = 0 - 2\lambda(y - R\sin\omega t) + F_{py}$$

$$m\ddot{z} = -gm - 2\lambda z + F_{pz}$$

Values of forces in particular coordinates

External forces

- for example drag

Lagrange multiplier

- needed to be evaluated

$$\phi(x, y, z) = (x - R\cos\omega t)^2 + (y - R\sin\omega t)^2 + z^2 - l^2 = 0$$

making derivatives

$$(x - R\cos\omega t)^2 + (y - R\sin\omega t) + z^2 = l^2$$

Let us derive λ first

modifying

$$2(x - R\cos\omega t)\,\dot{(x - R\cos\omega t)} + 2(y - R\sin\omega t)\dot{(y - R\sin\omega t)} + 2z\dot{z} = 2l\dot{l}$$

$$(x - R\cos\omega t)\dot{(x - R\cos\omega t)} + (y - R\sin\omega t)\dot{(y - R\sin\omega t)} + z\dot{z} = l\dot{l}$$

$$(x - R\cos\omega t)(\dot{x} + \omega R\sin\omega t) + (y - R\sin\omega t)(\dot{y} - \omega R\cos\omega t) + z\dot{z} = l\dot{l}$$

making a derivative once more

$$(\dot{x} + \omega R\sin\omega t)^2 + (\dot{y} - \omega R\cos\omega t)^2 + \dot{z}^2 + (x - R\cos\omega t)(\ddot{x} + \omega^2 R\cos\omega t) +$$

$$+(y - R\sin\omega t)(\ddot{y} + \omega^2 R\sin\omega t) + z\ddot{z} = \dot{l}\dot{l} + l\ddot{l}$$

modifying, marking part of the expression by „A“

$$\ddot{x}(x - R\cos\omega t) + \ddot{y}(y - R\sin\omega t) + \ddot{z}z = \dot{l}\dot{l} + l\ddot{l} - (\dot{x} + \omega R\sin\omega t)^2 - (\dot{y} - \omega R\cos\omega t)^2 -$$

$$-\dot{z}^2 - \omega^2 R\cos\omega t(x - R\cos\omega t) - \omega^2 R\sin\omega t(y - R\sin\omega t) = \dot{l}\dot{l} + l\ddot{l} + A$$

..now „A“

$$\ddot{x}(x - Rcos\omega t) + \ddot{y}(y - Rsin\omega t) + \ddot{z}z = \dot{l}\dot{l} + l\ddot{l} - (\dot{x} + \omega Rsin\omega t)^2 - (\dot{y} - \omega Rcos\omega t)^2 -$$

$$-\dot{z}^2 - \omega^2 Rcos\omega t(x - Rcos\omega t) - \omega^2 Rsin\omega t(y - Rsin\omega t) = \dot{l}\dot{l} + l\ddot{l} + A$$

Defining „A“

$$A = -(\dot{x} + \omega Rsin\omega t)^2 - (\dot{y} - \omega Rcos\omega t)^2 - \dot{z}^2 - \omega^2 R[cos\omega t(x - Rcos\omega t) + sin\omega t(y - Rsin\omega t)]$$

$$\ddot{x} = 0 - \frac{\lambda}{m}2(x - Rcos\omega t)$$

$$\ddot{y} = 0 - \frac{\lambda}{m}2(y - Rsin\omega t)$$

$$\ddot{z} = -g - \frac{\lambda}{m}2z$$

Accelerations in particular coordinates

$$\ddot{x}(x - Rcos\omega t) = -\frac{2\lambda}{m}(x - Rcos\omega t)^2$$

$$\ddot{y}(y - Rsin\omega t) = -\frac{2\lambda}{m}(y - Rsin\omega t)^2$$

$$(\ddot{z} + g)z = -\frac{2\lambda}{m}z^2$$

..continuing with λ

$$\ddot{x}(x - R\cos\omega t) = -\frac{2\lambda}{m}(x - R\cos\omega t)^2$$

$$\ddot{y}(y - R\sin\omega t) = -\frac{2\lambda}{m}(y - R\sin\omega t)^2$$

$$(\ddot{z} + g)z = -\frac{2\lambda}{m}z^2$$

addition

$$\ddot{x}(x - R\cos\omega t) + \ddot{y}(y - R\sin\omega t) + (\ddot{z} + g)z = -\frac{2\lambda}{m}(x - R\cos\omega t)^2 + \frac{2\lambda}{m}(y - R\sin\omega t)^2 + \frac{2\lambda}{m}z^2$$

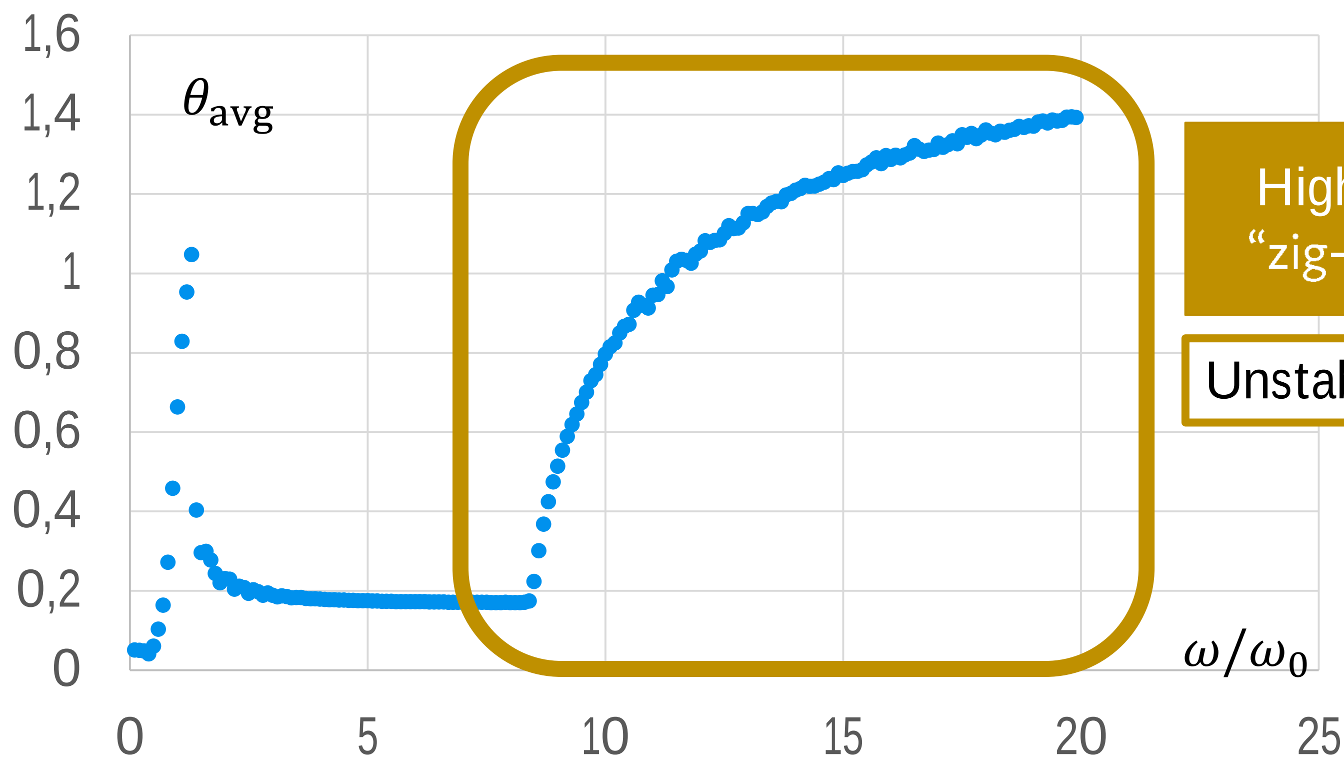
modifying

$$\ddot{x}(x - R\cos\omega t) + \ddot{y}(y - R\sin\omega t) + (\ddot{z} + g)z = -\frac{2\lambda}{m}\left[(x - R\cos\omega t)^2 + (y - R\sin\omega t)^2 + z^2\right]$$

l^2

$$\ddot{x}(x - R\cos\omega t) + \ddot{y}(y - R\sin\omega t) + (\ddot{z} + g)z = -\frac{2\lambda}{m}l^2$$

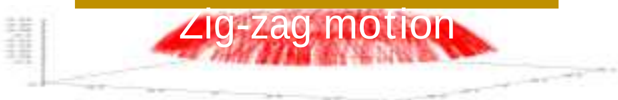
Unstable motion



Highly-unstable "zig-zag" motion

Unstable in φ angle

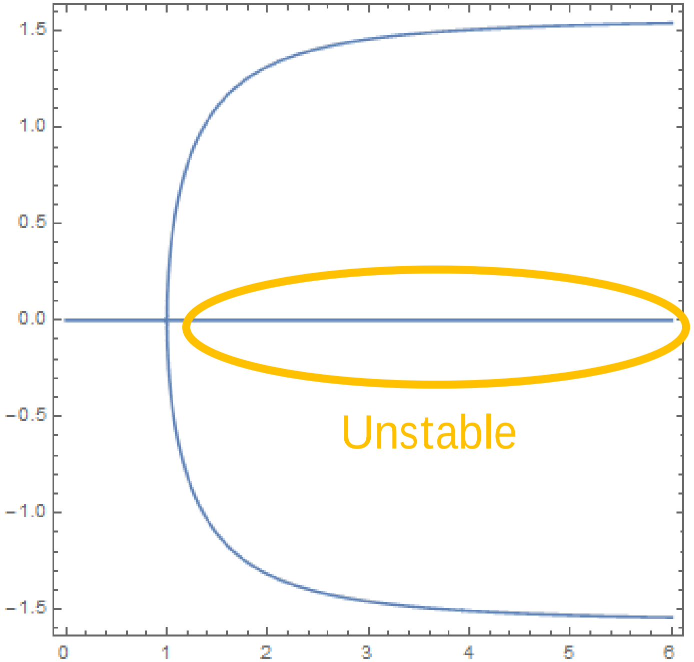
Zig-zag motion



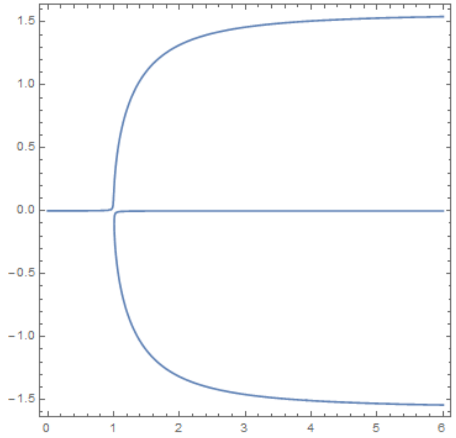
View from outside

Variation on textbook problem

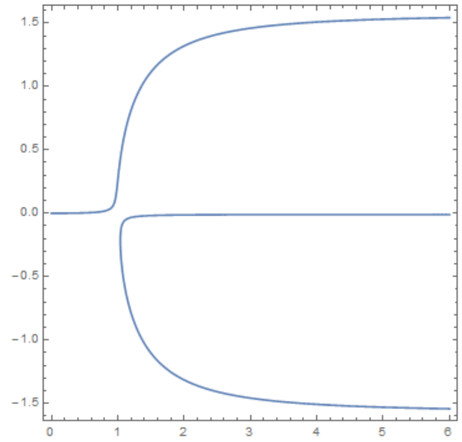
Special case: $R = 0 \quad \Rightarrow$



$R/L = 0$

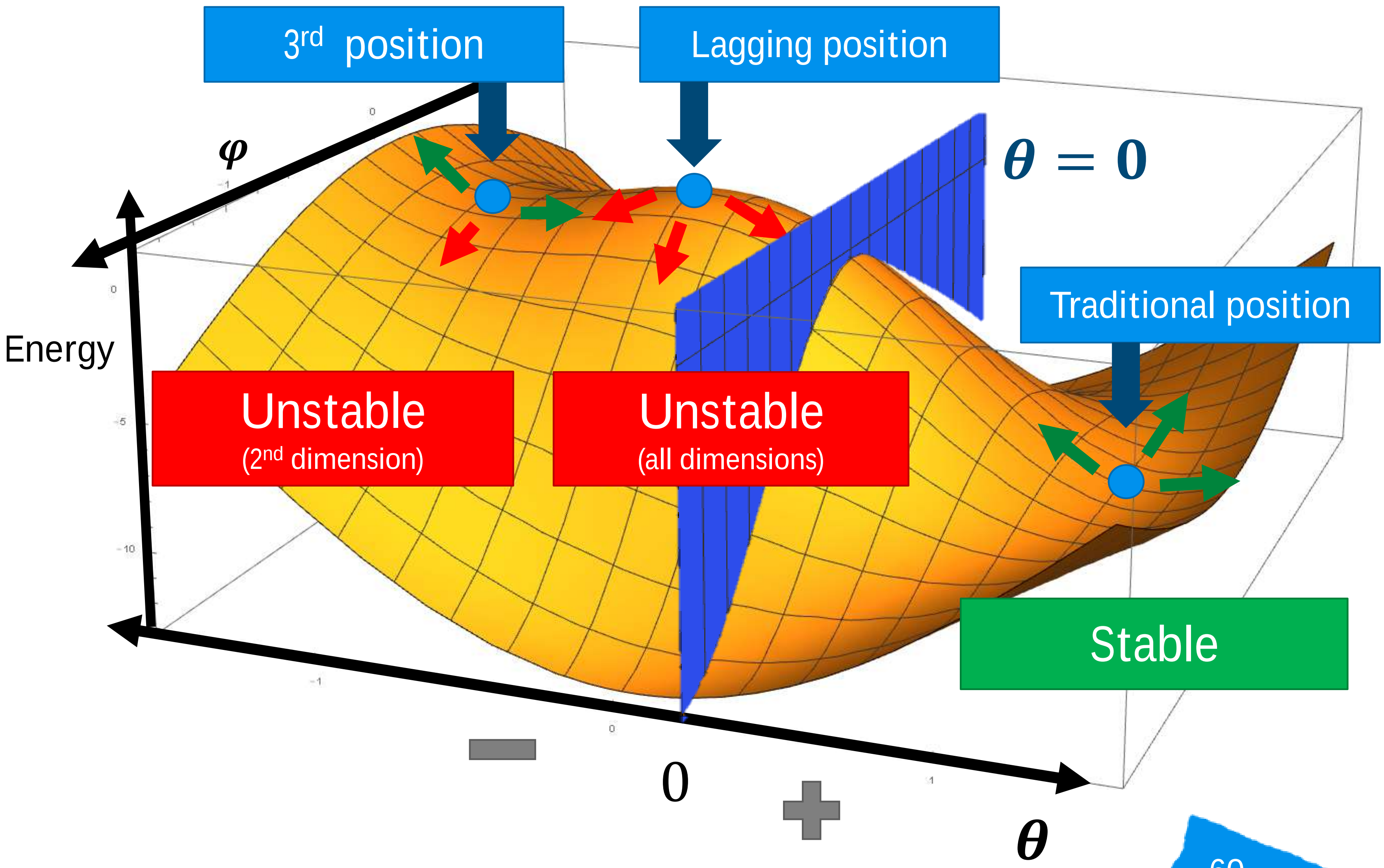


$R/L = 0{,}001$



$R/L = 0{,}01$

3D Energy plot



Stability analysis of found positions

Total potential energy in the reference frame of arm

[Arbitrary units]

$L = 0{,}3m$
$R = 0{,}1m$
$\omega = 10rad.s^{-1}$