1. Invent Yourself

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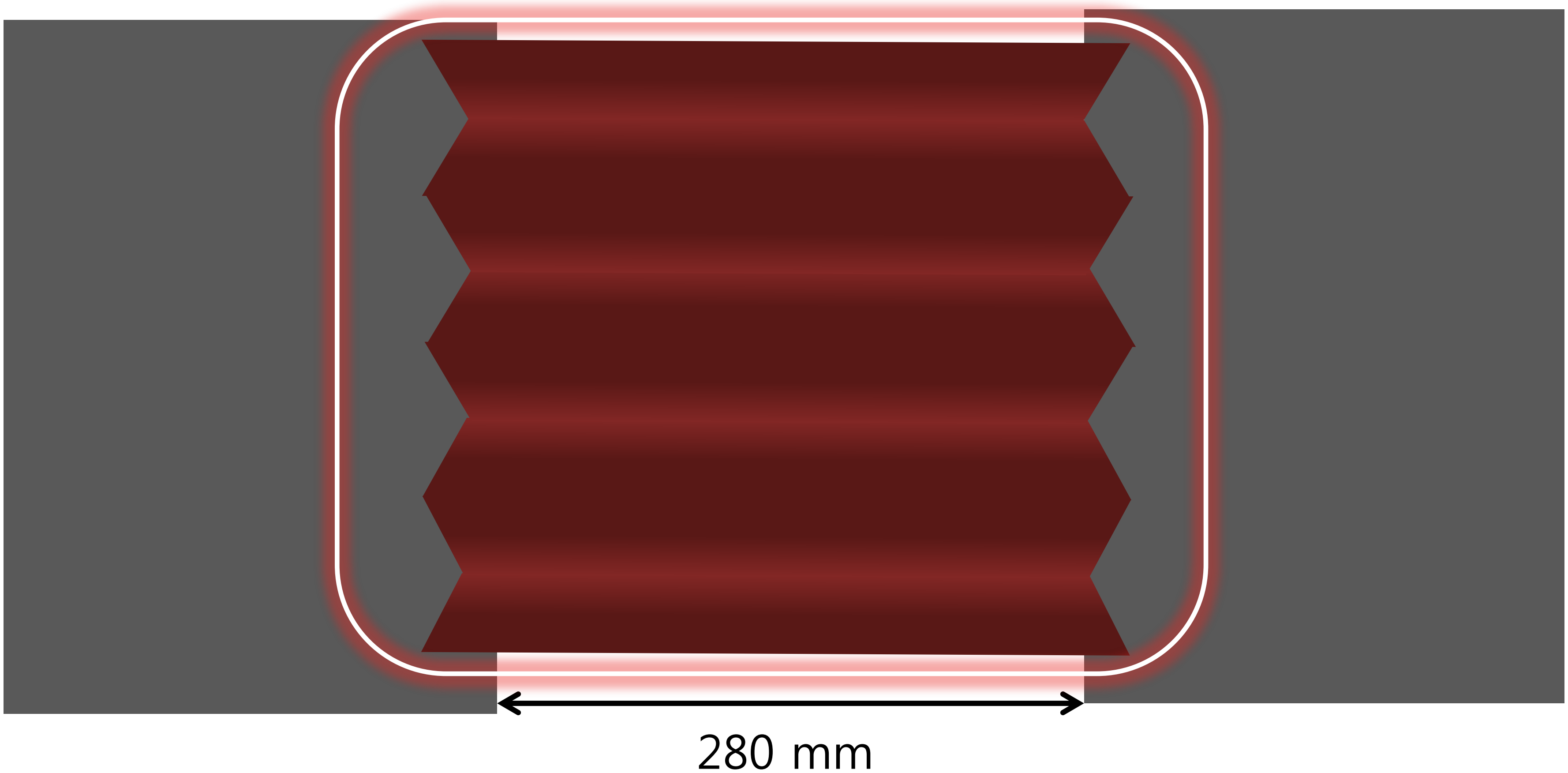
Team Korea

Problem Statement



Introduce ***parameters*** to describe the ***strength*** of your bridge, and ***optimise*** some or all of them.

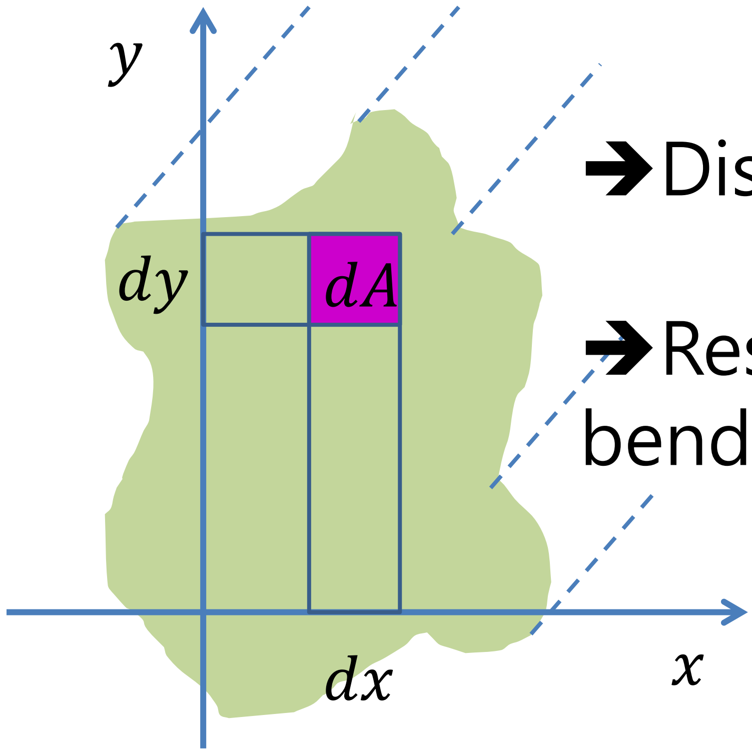
Problem Statement

It is ***more difficult to bend*** a paper sheet, if it is folded "***accordion style***" or ***rolled into a tube***.

Why?

→Increased **second moment of area**

Second Moment of Area



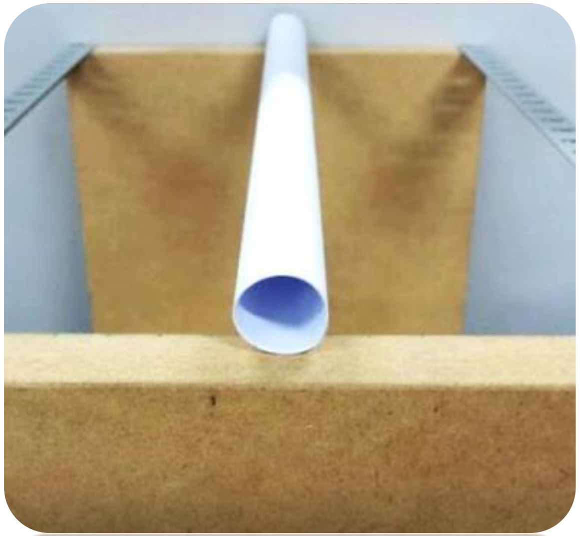
➔Distribution of Area

➔Resistance of a beam to bending

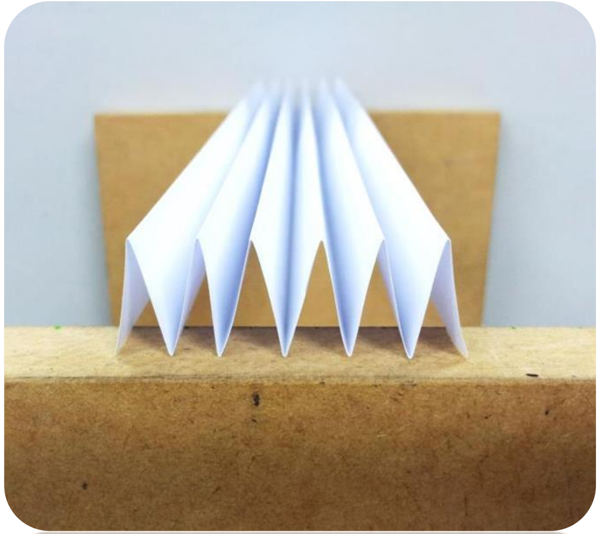
$$I_x = \int\int_A y^2 dxdy$$

Second Moment of Area of Paper Bridge

n : number of layers/
number of bumps
θ : contact angle



Length(L)=297m
m



Width(D)=210mm

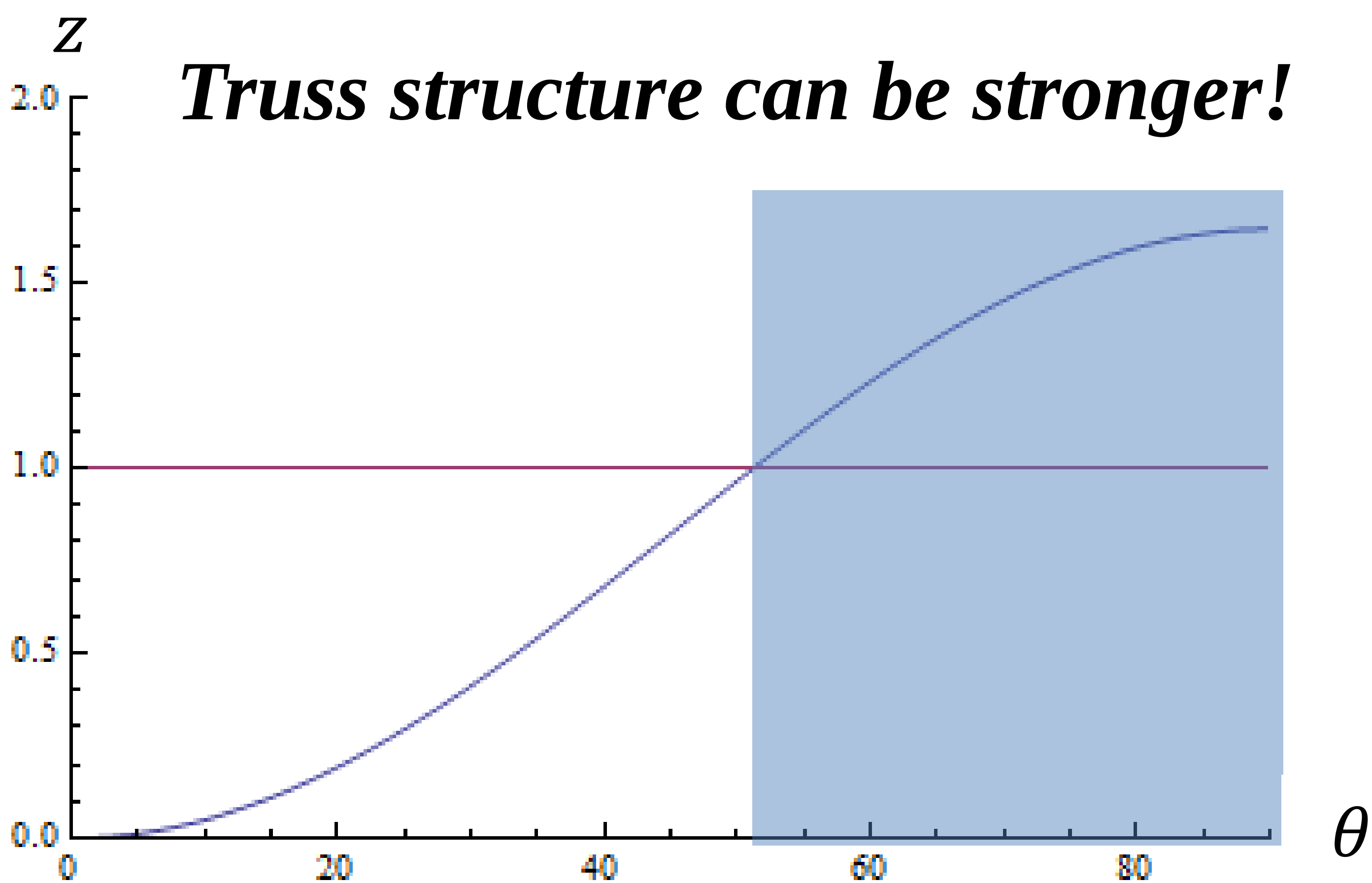
Thickness(τ)=0.14m
m

$$I_{z.tube} = \frac{D^3\tau}{8\pi^2 n^2}$$

$$I_{z.truss} = \frac{D^3\tau}{96n^2}(1 - \cos 2\theta)$$

Second Moment of Area (Truss vs Tube)

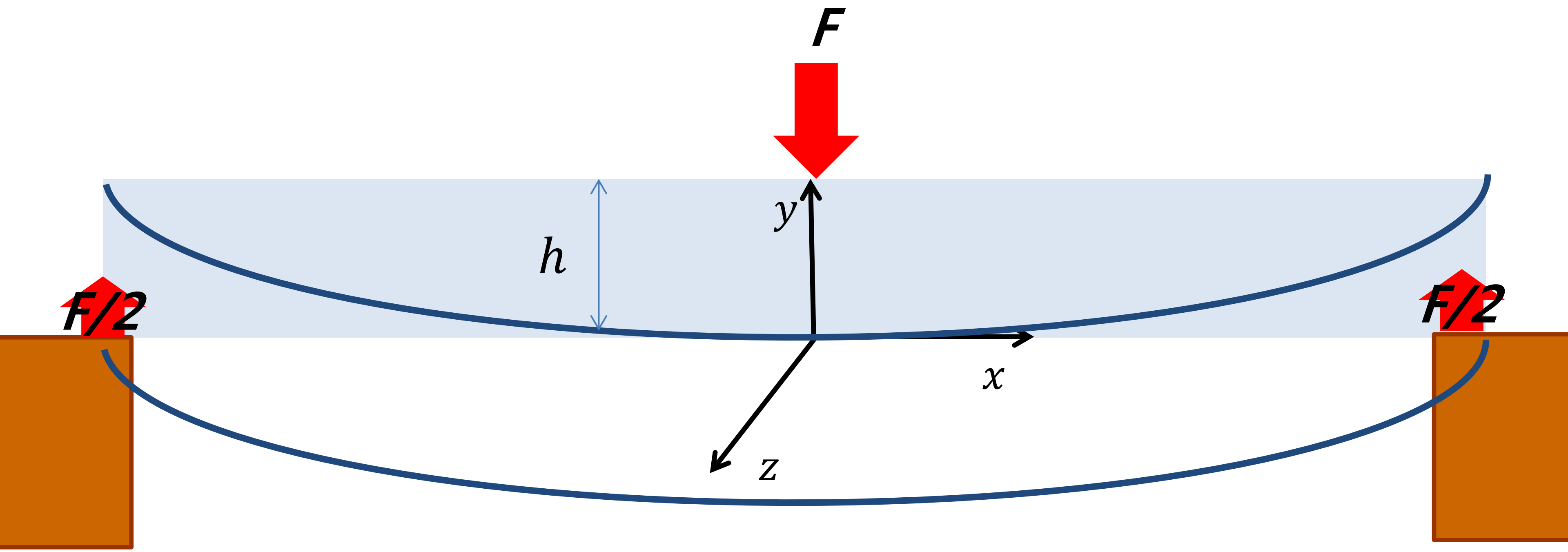
Truss structure can be stronger!



$$z = \frac{I_{z.truss}}{I_{z.tube}} = \frac{\pi^2(1 - cos2\theta)}{12}$$

$*$ θ: contact angle in truss structure

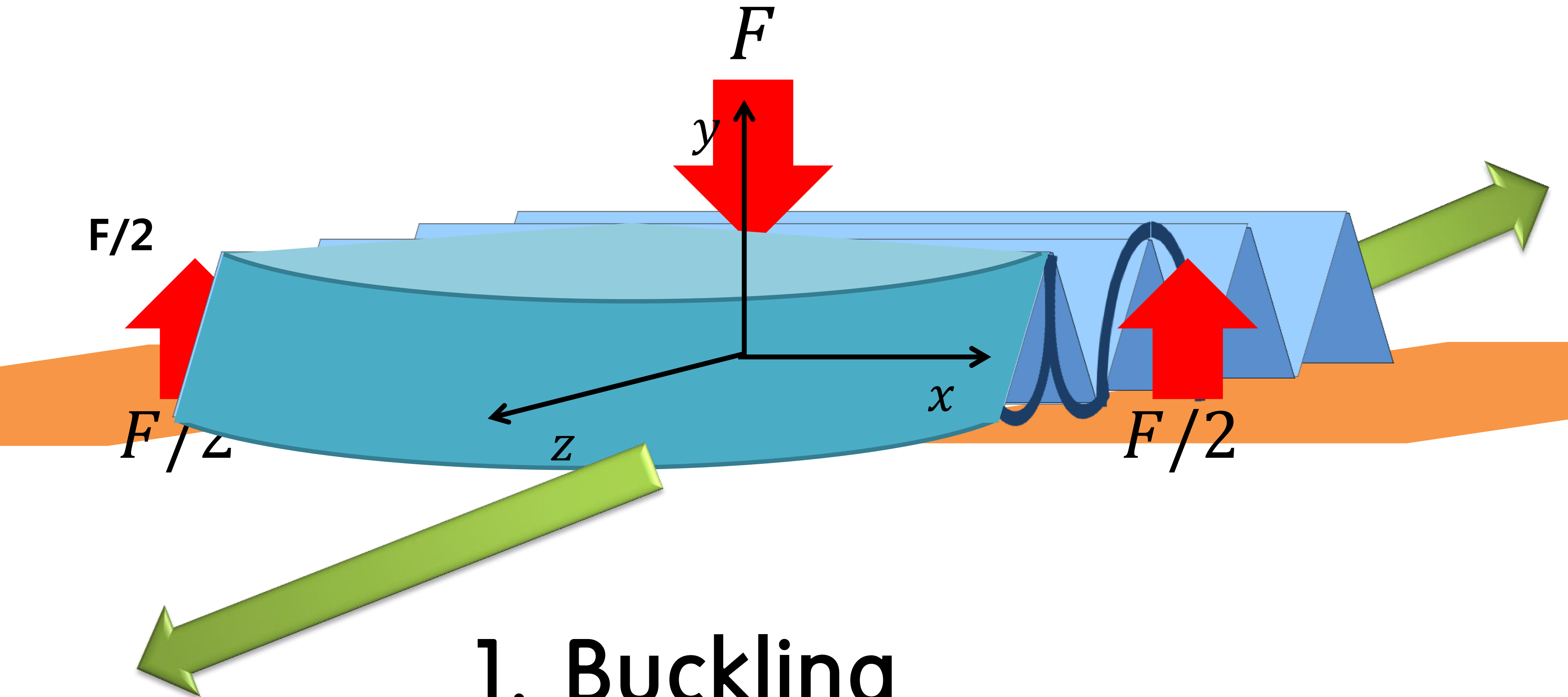
What is “collapse”?



$$\Delta y = -h \rightarrow \text{Collapse}$$

Maximum Mass = Strength

How does a bridge collapse?

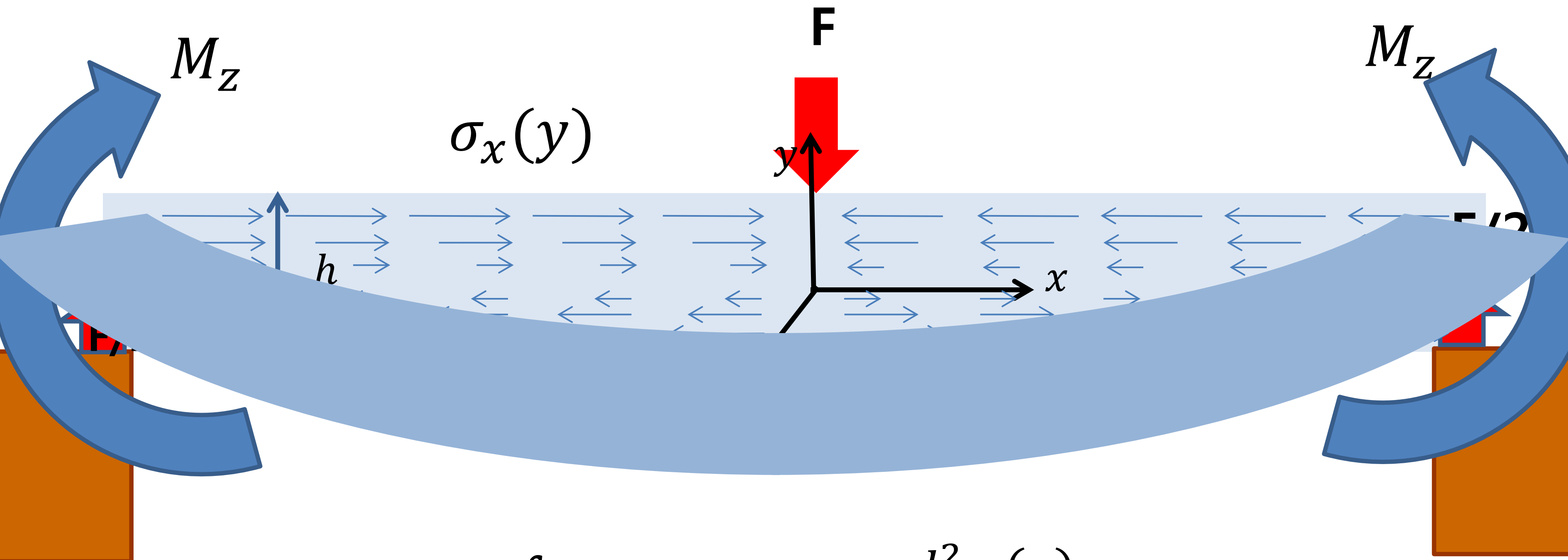


1. Buckling

2. Sliding

3. Necking

Shear Force and Bending Moment



$$M_z = \int_A \sigma_x y \, dA = -EI \frac{d^2 w(x)}{dx^2}$$

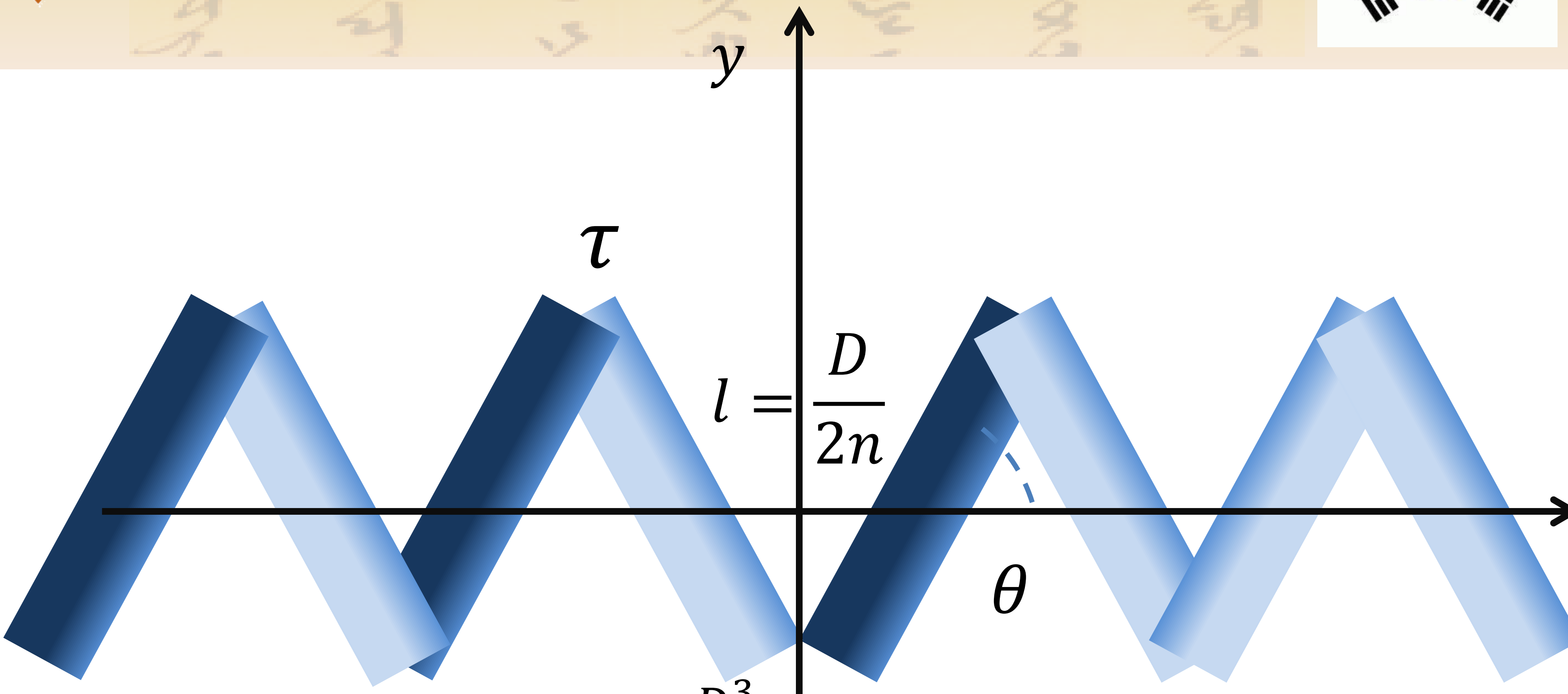
$$w(0) = -\frac{FL^3}{48EI}$$

$$Collapse : w(0) = -h$$

$$\boldsymbol{F_{max} = \frac{48EIh}{L^3}}$$

M_z: Bending Moment
E : Young's Modulus
I: Second Moment of Area
w : Deflection of the axis of the beam

Second Moment of Area



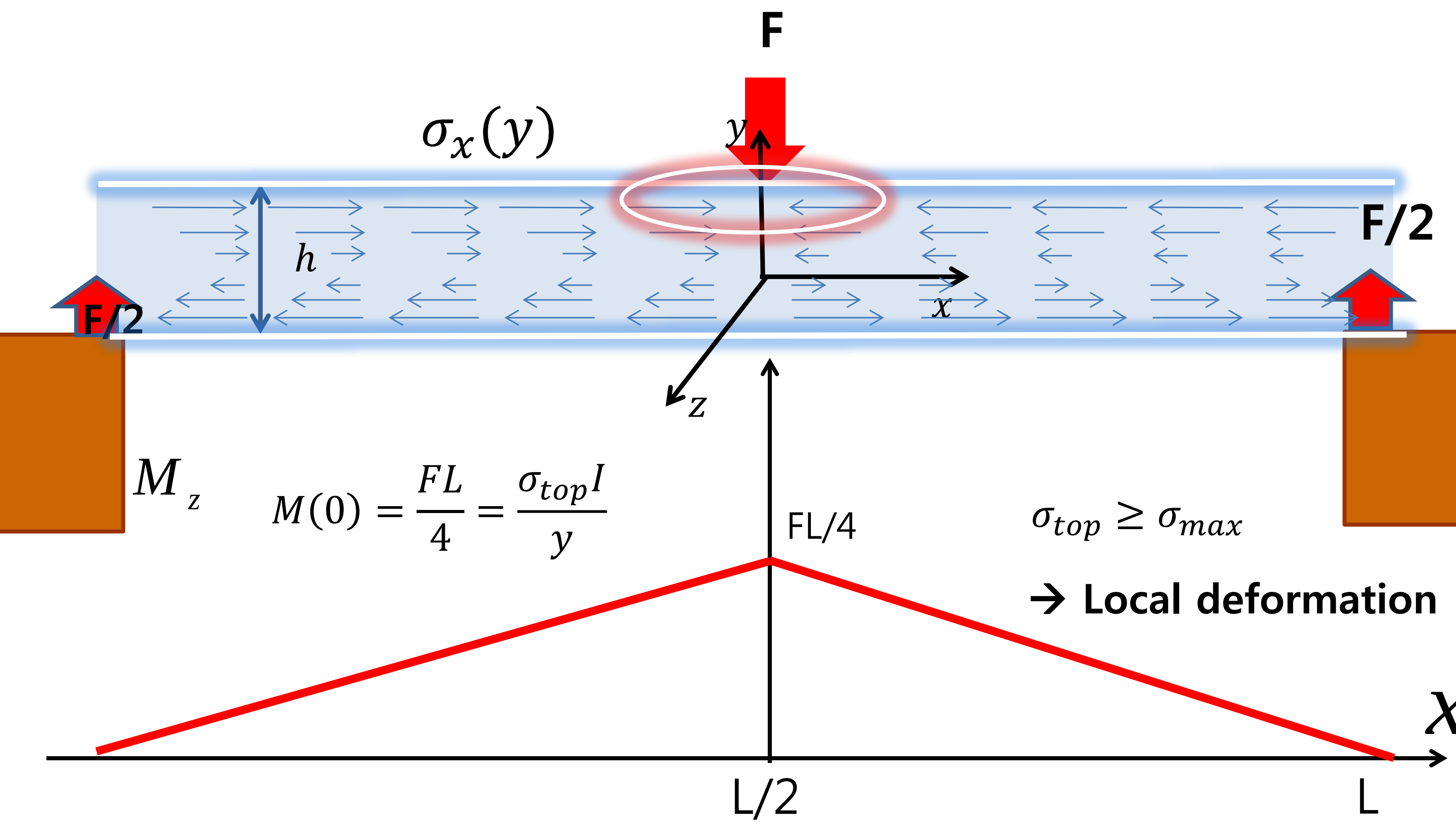
$$I_z = \frac{D^3\tau}{96n^2}(1 - \cos 2\theta)$$

$$\boldsymbol{F_{max}} = \frac{\boldsymbol{ED^4\tau sin^3\theta}}{\boldsymbol{2L^3n^3}} \approx \frac{\boldsymbol{2.22 \times 10^4 sin^3\theta}}{\boldsymbol{n^3}}(\mathrm{N})$$

M. Ahmer Wadee, G. W. Hunt M. A. Peletier Kink Band Instability in Layered Structures

Local Deformation

Local Deformation



$$M(0) = \frac{FL}{4} = \frac{\sigma_{top} I}{y}$$

$$\sigma_{top} \geq \sigma_{max}$$

→ Local deformation

* σ_{max} : Ultimate Compressive Stress

Local Deformation

$$M(0) = \frac{FL}{4} = \frac{\sigma_{top} I}{y} \qquad \sigma_{top} \geq \sigma_{max}$$

$$F_{max} = \frac{4\sigma_{max} I}{Ly}$$

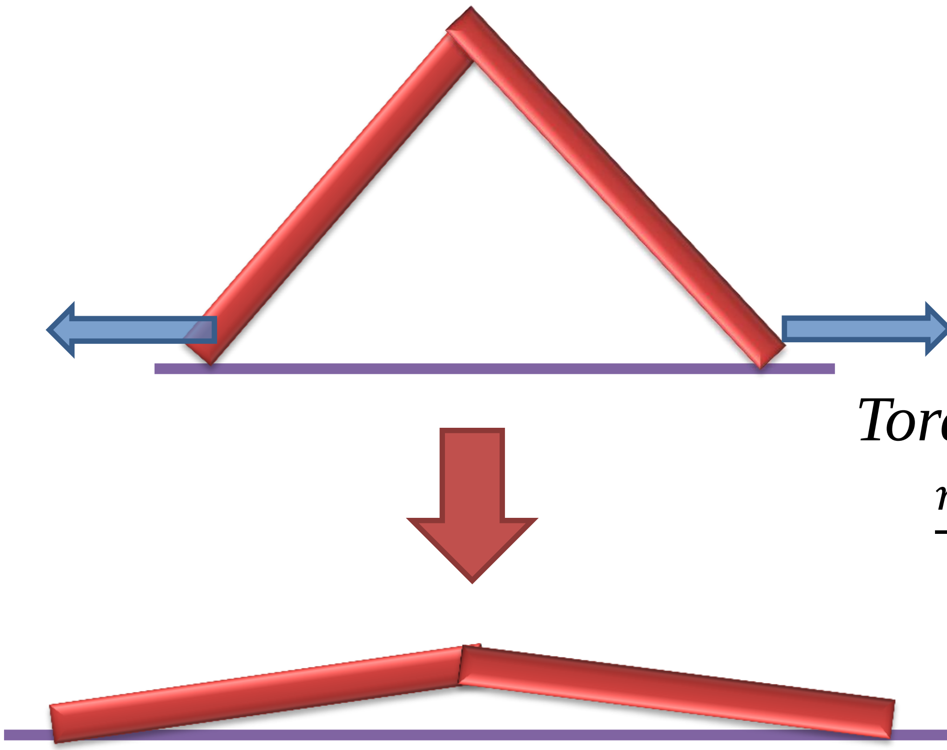
$$y = \frac{h}{2} = \frac{D \sin\theta}{2n} \qquad I_z = \frac{D^3 \tau}{96 n^2}(1 - \cos 2\theta)$$

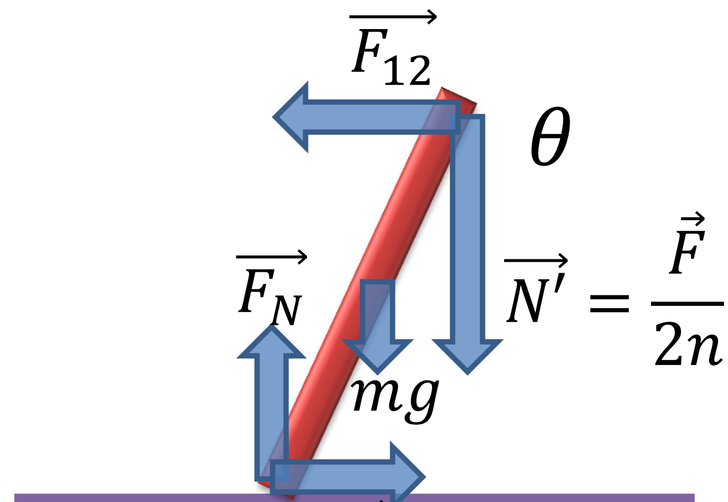
$$F_{max} = \frac{\sigma_{max} D^2 \tau \sin\theta}{6nL}$$

$$10 \quad GPa \leq \sigma_{max} \leq 100 GPa \qquad 34.6\text{g} \leq \frac{m_{max} n}{\sin\theta} \leq 346\text{g}$$

M. Ahmer Wadee, G. W. Hunt M. A. Peletier Kink Band Instability in Layered Structures

Second Scenario - Sliding





Torque Equilibrium

$$\frac{m_{tot.}g}{2n}\cos\theta\,\frac{l}{2} + \frac{F}{2n}\cos\theta\, l - f\,l\sin\theta = 0$$

$$f = \frac{\cot\theta}{2n}\left(F + \frac{m_{tot.}g}{2}\right) \approx \frac{\cot\theta\, F}{2n}$$

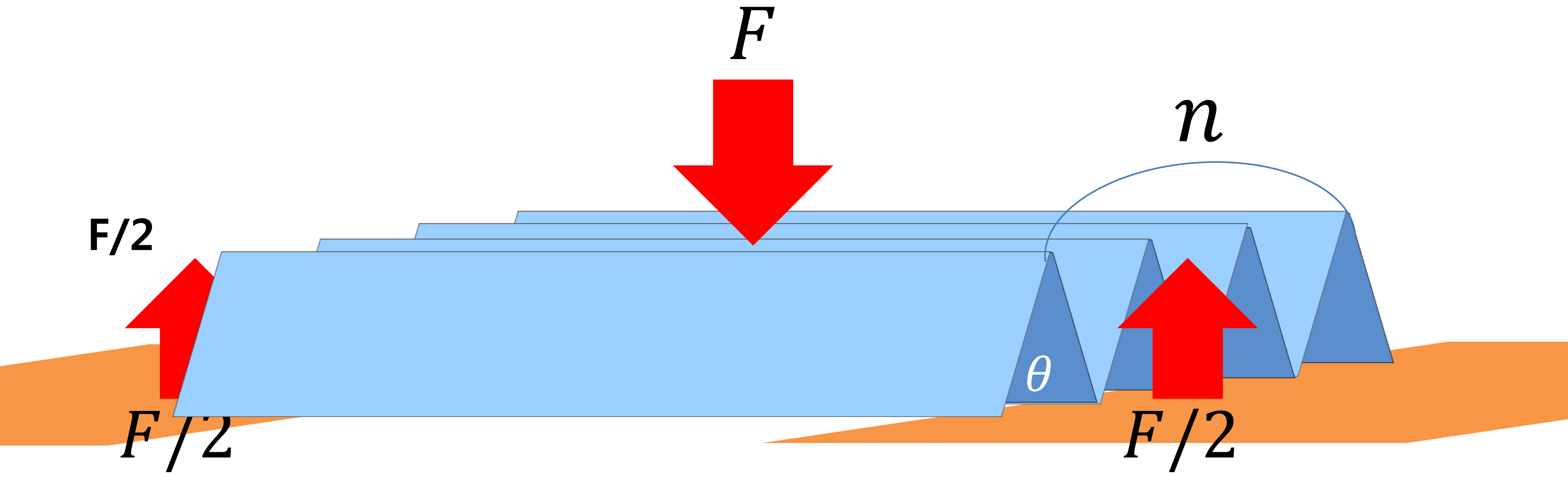
μ: frictional coefficient

Sliding Condition $\quad f = \mu F_N$

$$f < F_{12}$$

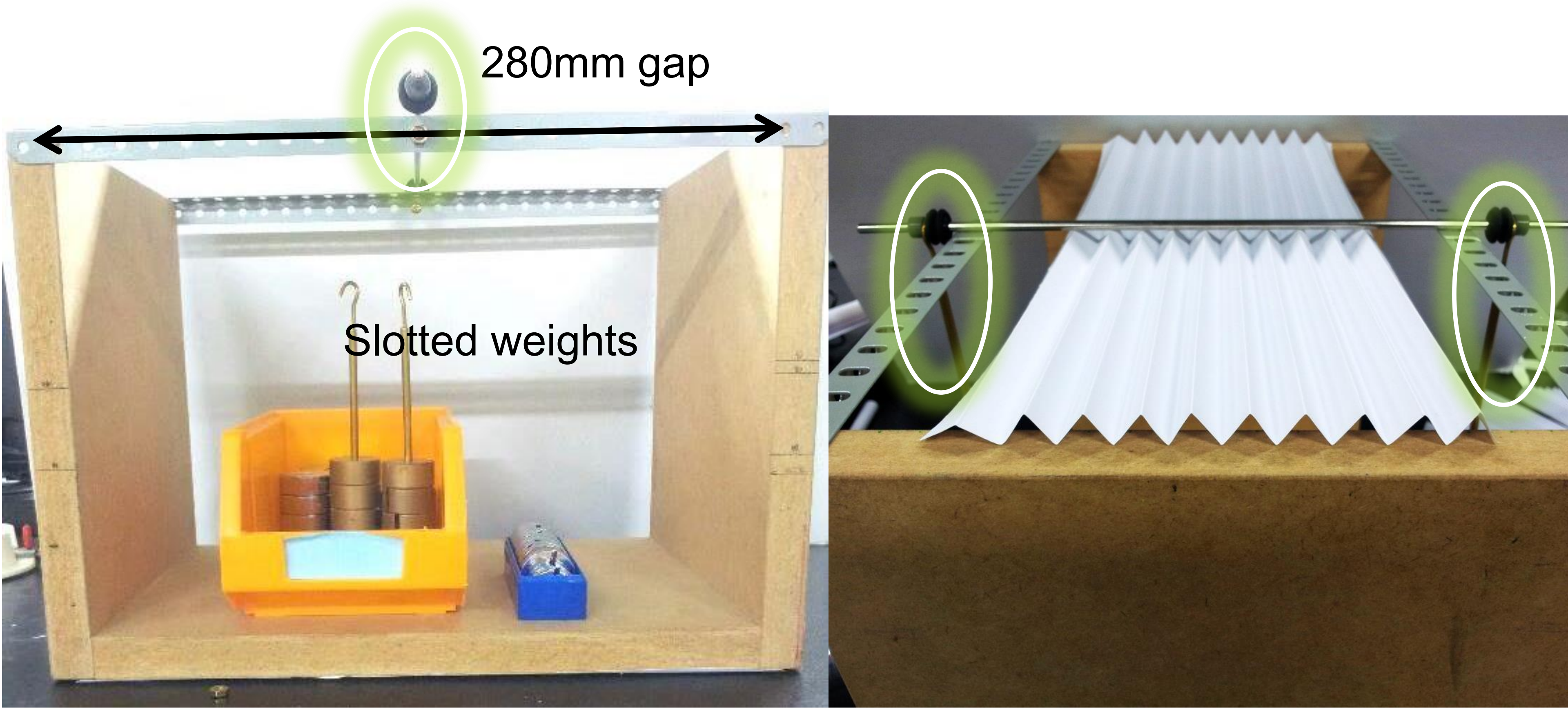
$$\therefore \tan^{-1}\frac{1}{\mu} \geq \theta \rightarrow \text{slide!}$$

Strength of the Bridge

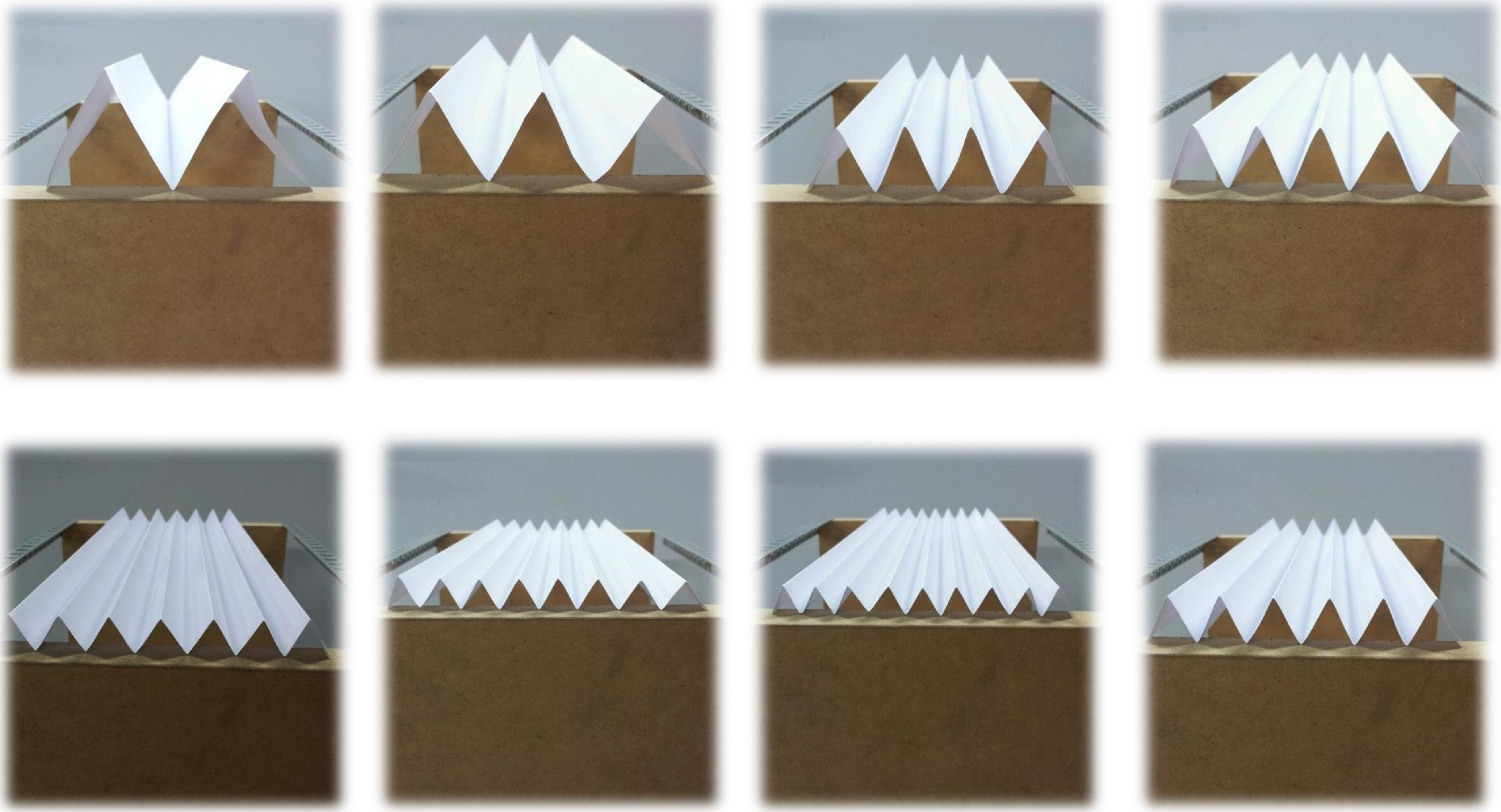


1. Buckling
2. Sliding

Experimental Setup

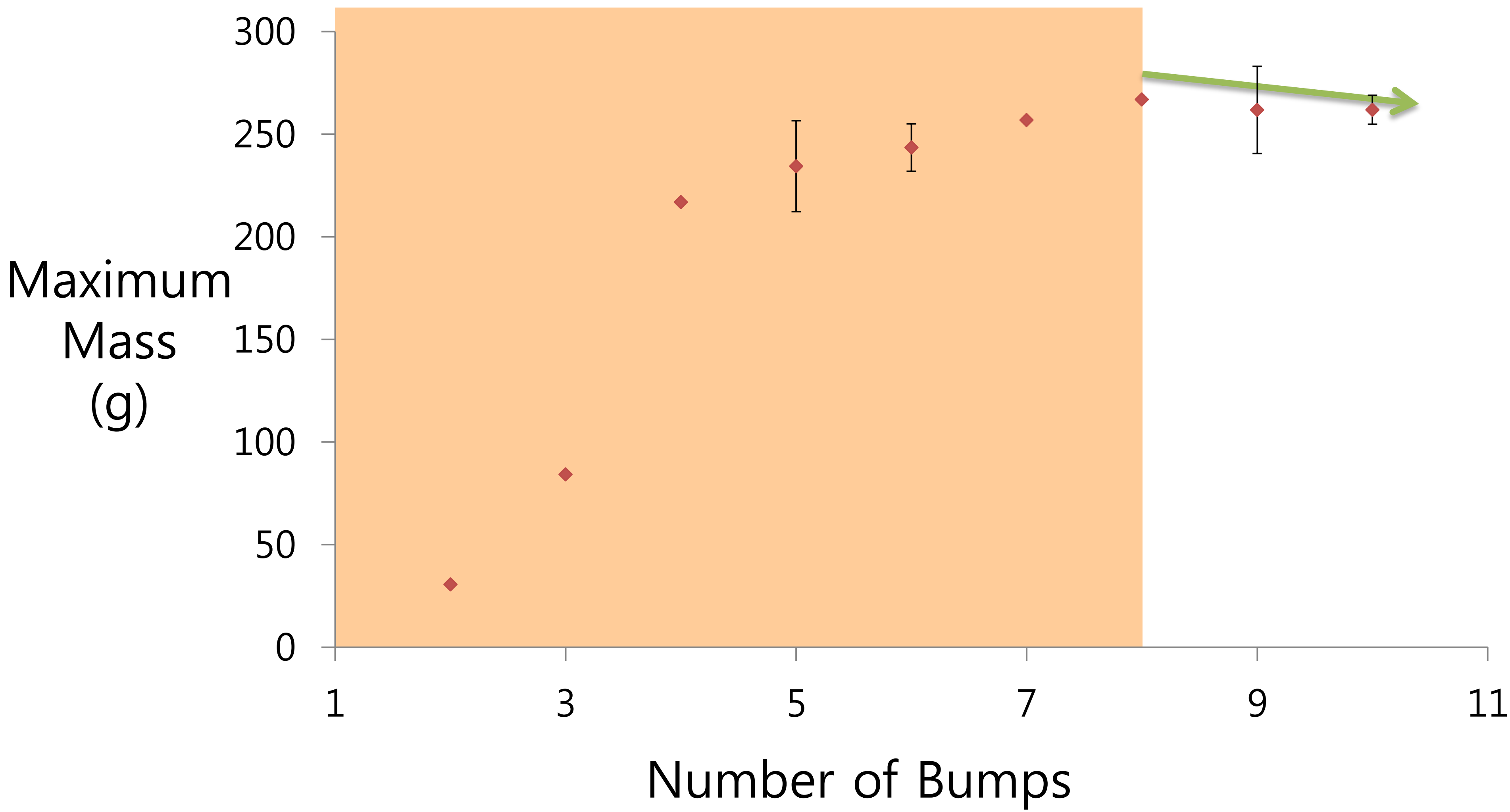


Number of Bumps

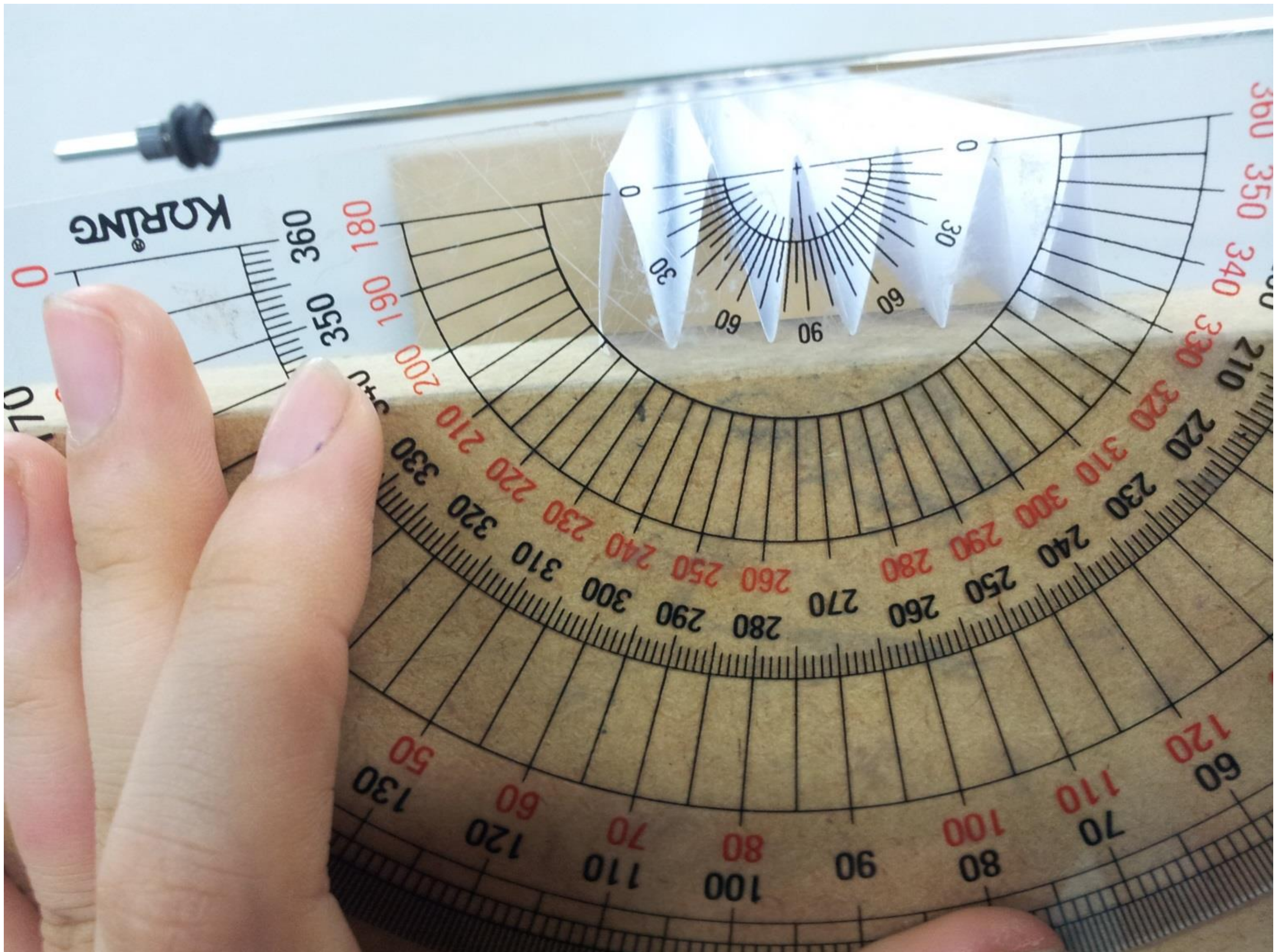


Contact Angle 50°

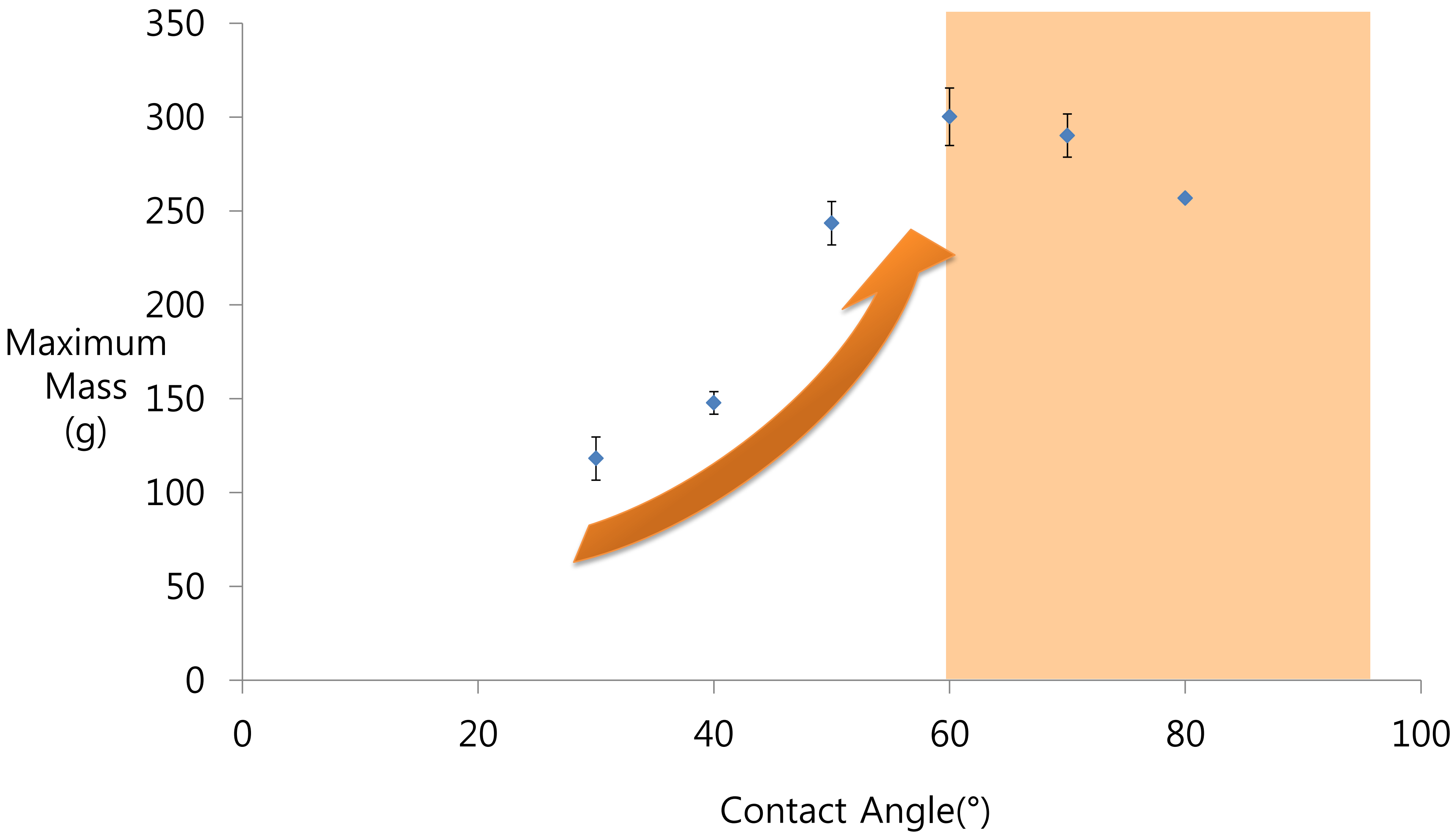
Number of Bumps vs Strength



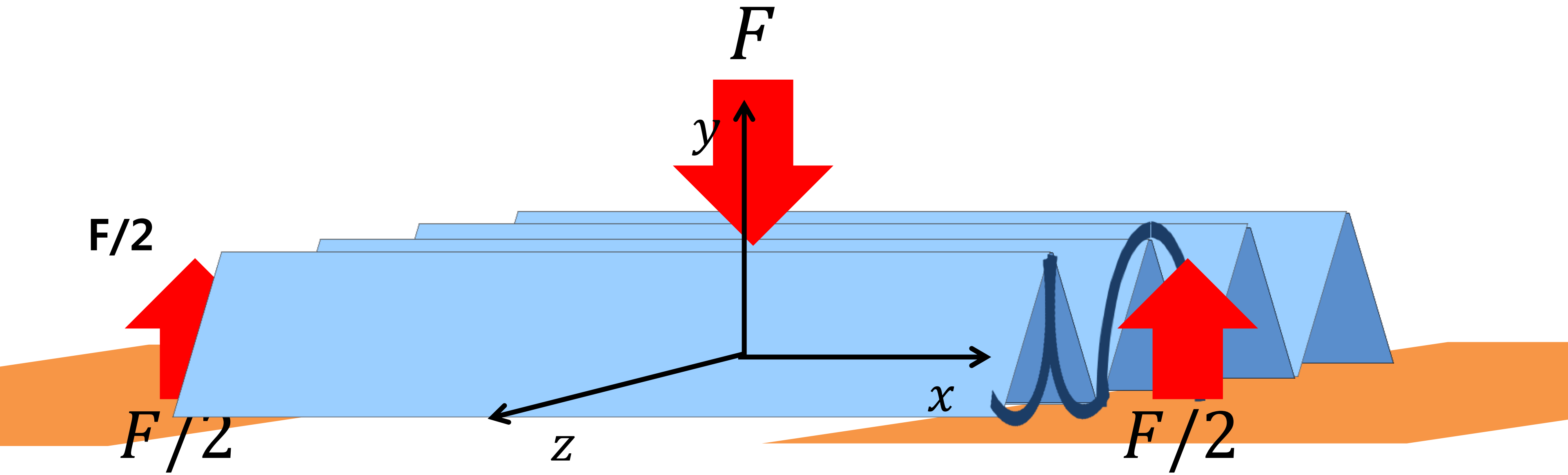
Angle(n=6)



Angle vs Strength(n=6)



Third Scenario-Necking

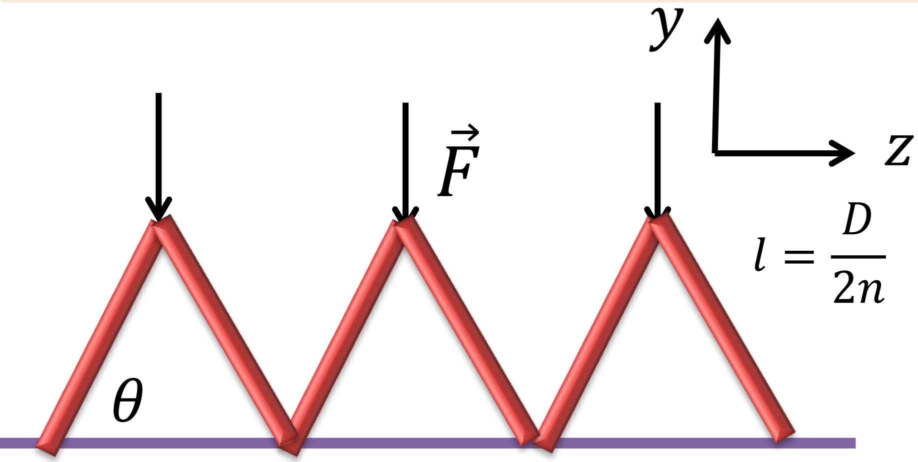


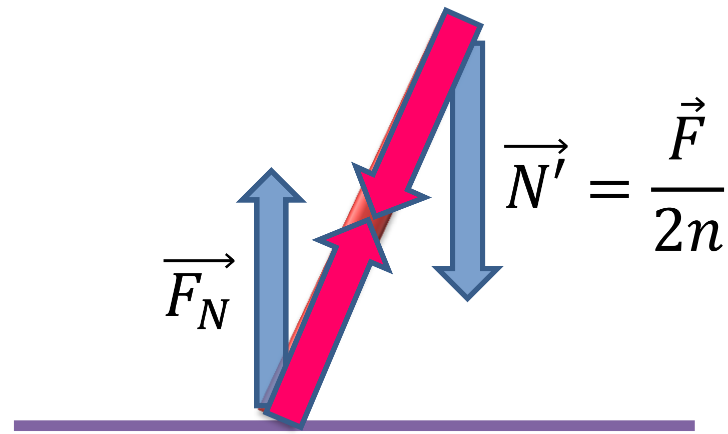
1. Buckling

2. Sliding

3. Necking

Third Scenario - Necking





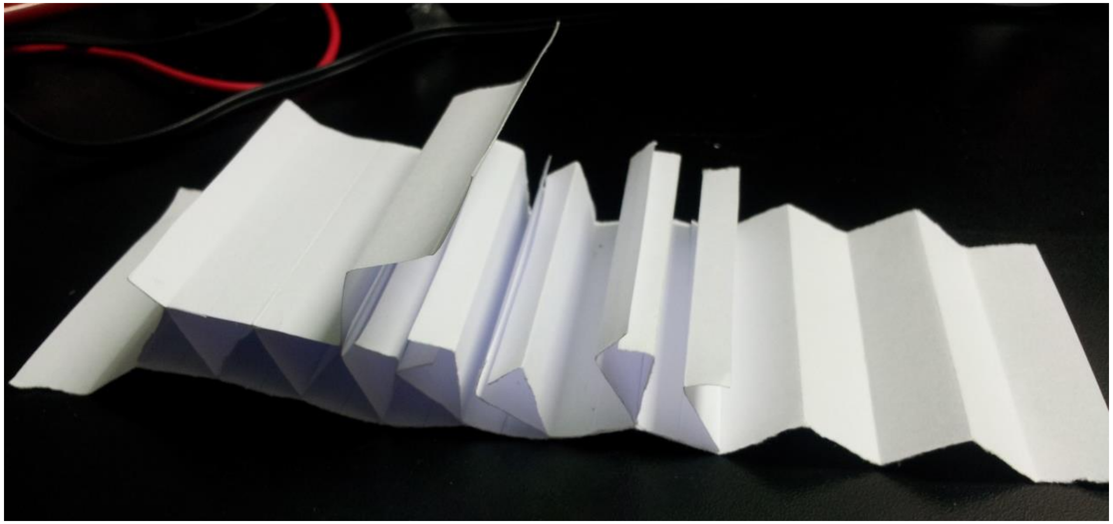
$$\frac{N' \sin\theta}{A} = \frac{F \sin\theta}{2n\tau L'} \geq \sigma_{max}$$

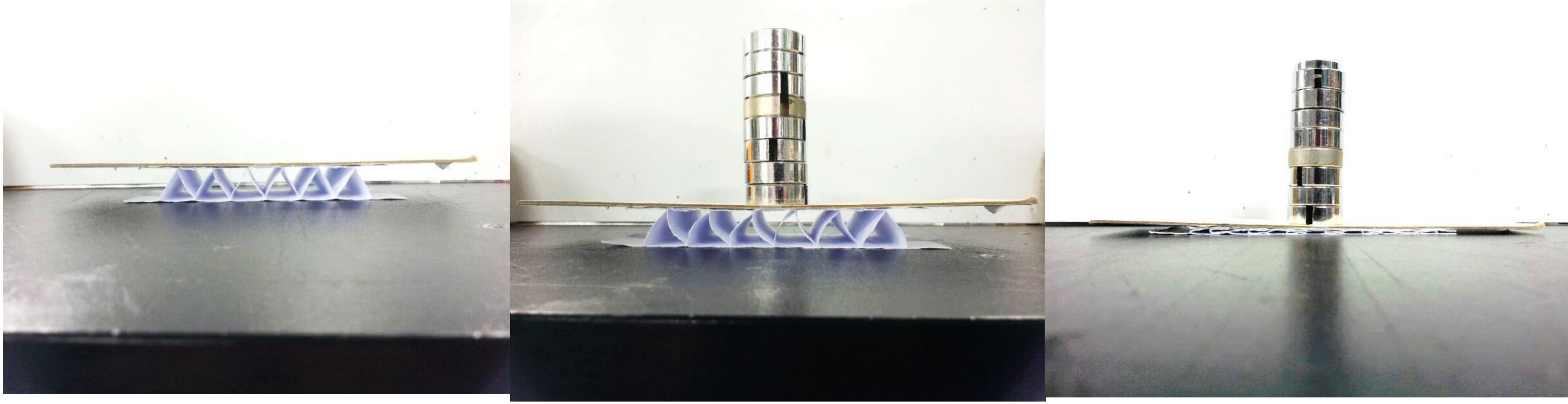
→ **Deflection of bridge member**

Ultimate Compressive Stress

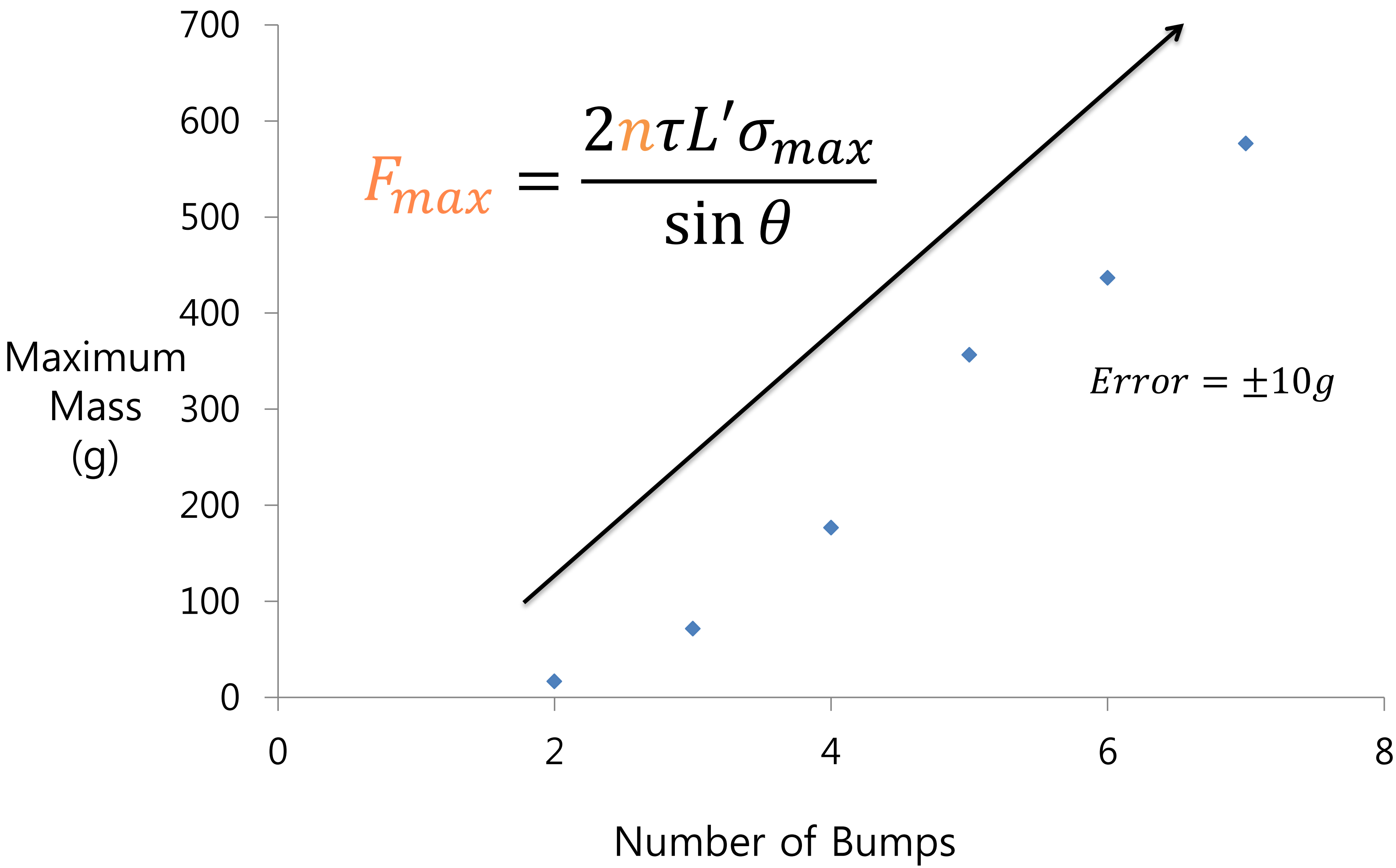
$$F_{max} = \frac{2n\tau L' \sigma_{max}}{\sin\theta}$$

Experiment-Necking Effect

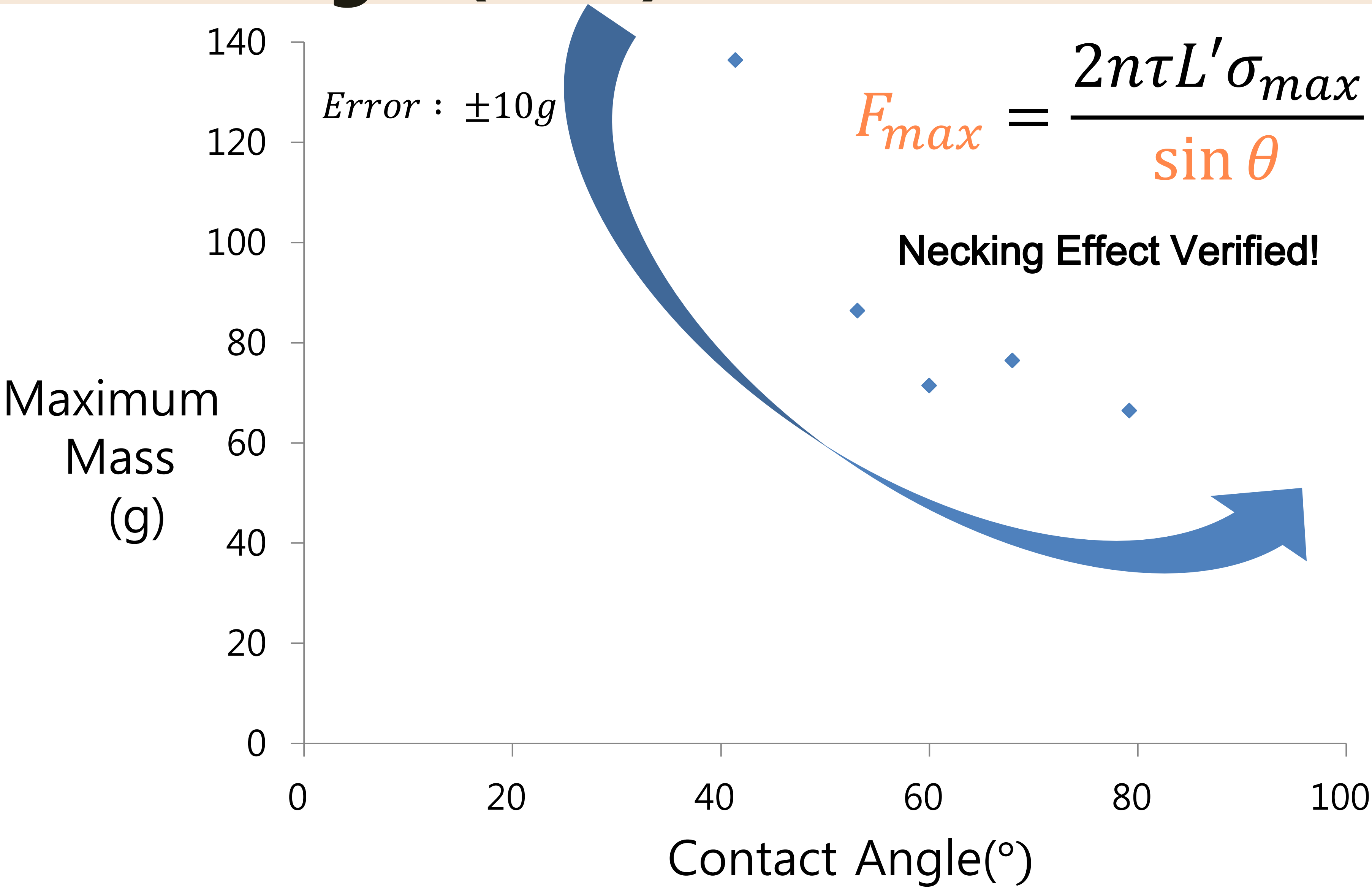




Number of Bumps vs Strength($\theta = 60°$)

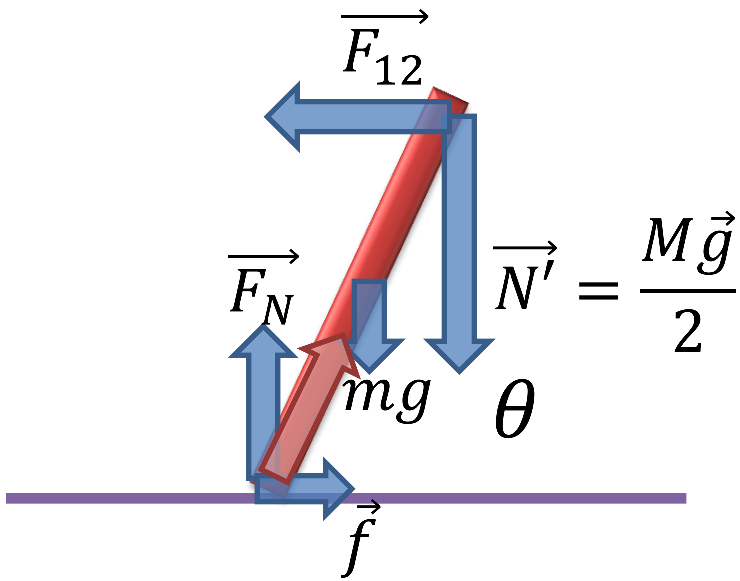


Contact Angle vs Strength(n=3)



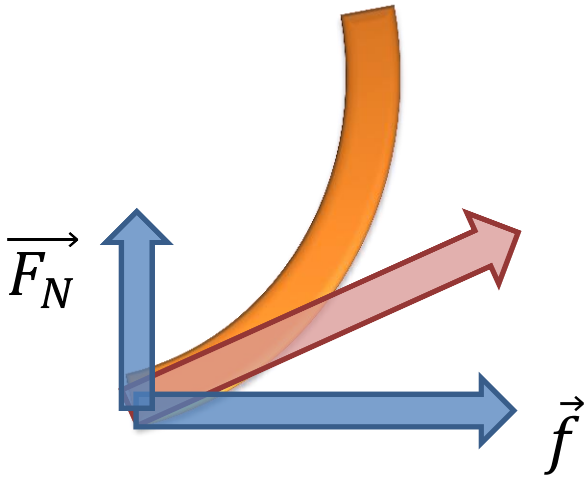
$$F_{max} = \frac{2n\tau L'\sigma_{max}}{\sin\theta}$$

Necking + Sliding



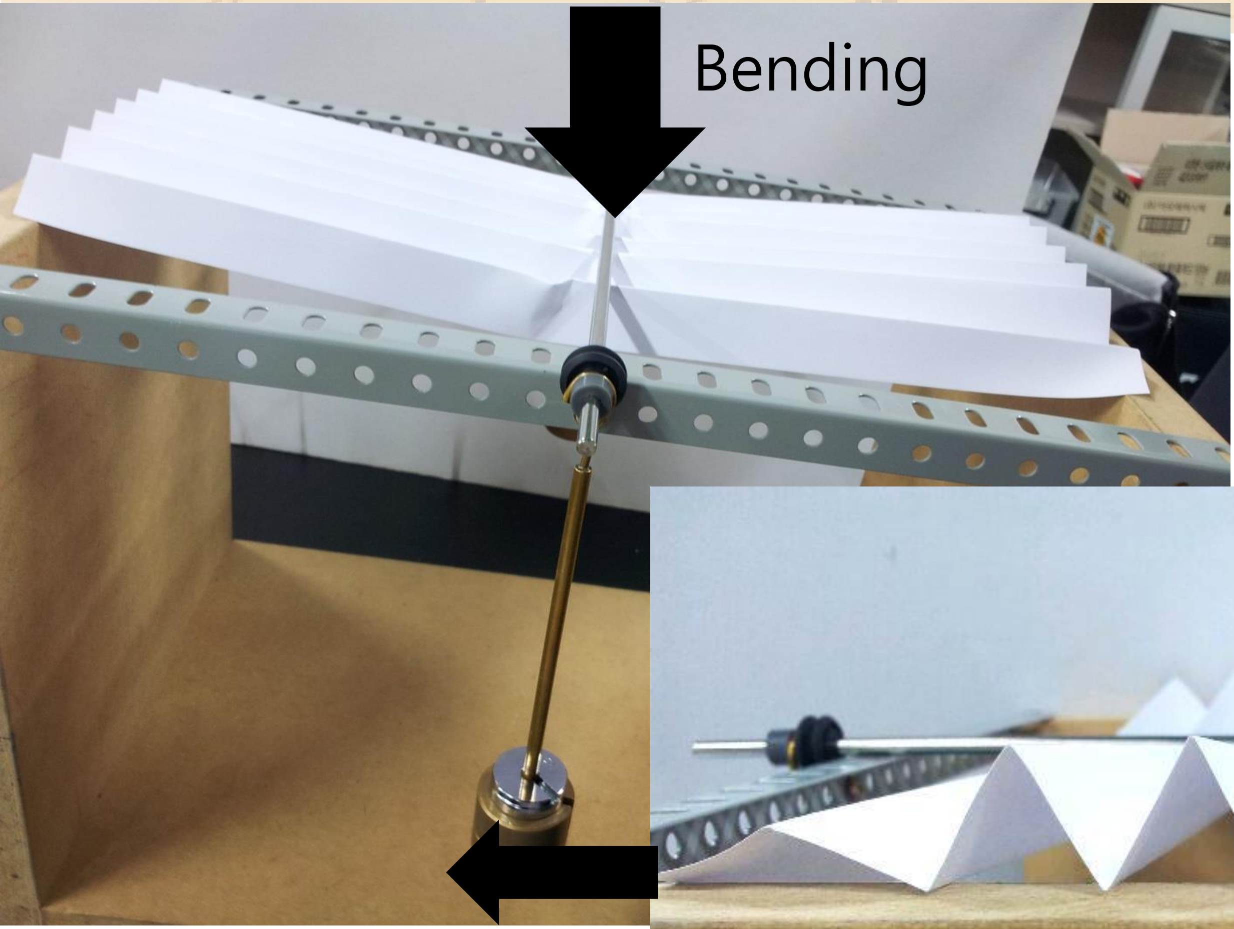
$$\tan^{-1}\frac{1}{\mu} \geq \theta$$

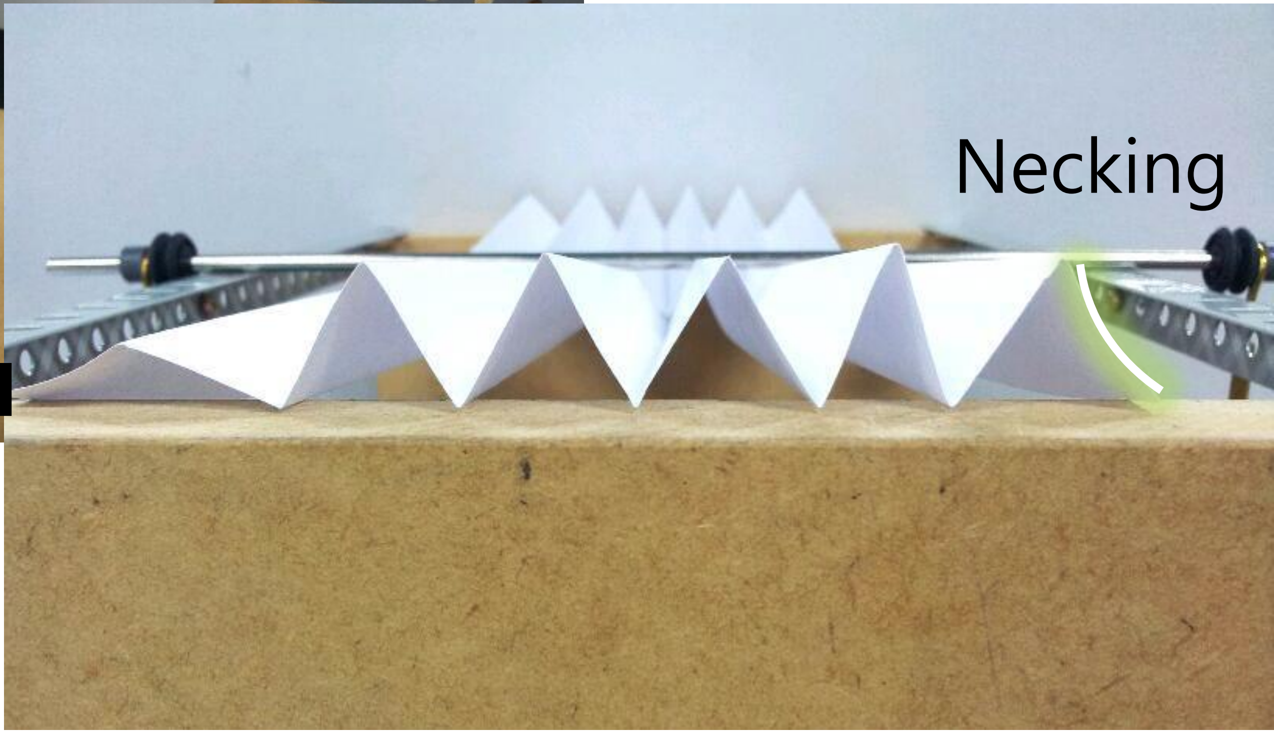
→ Slide!



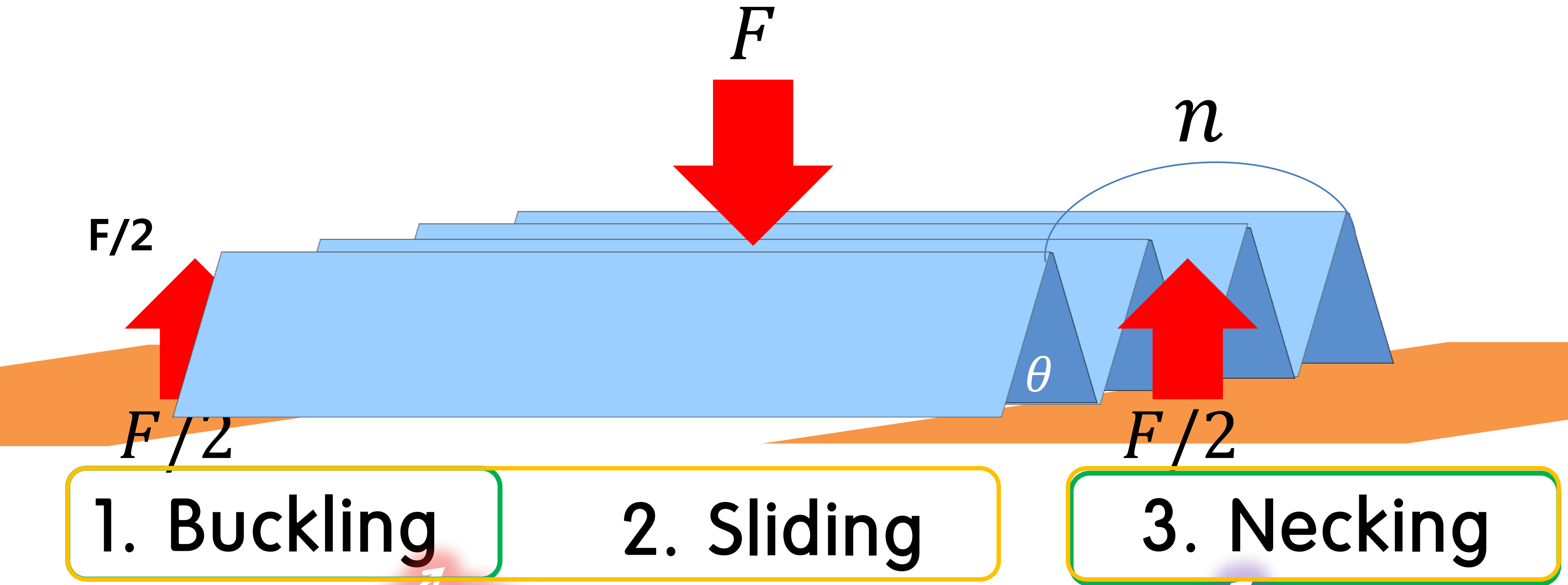
$\theta \downarrow$

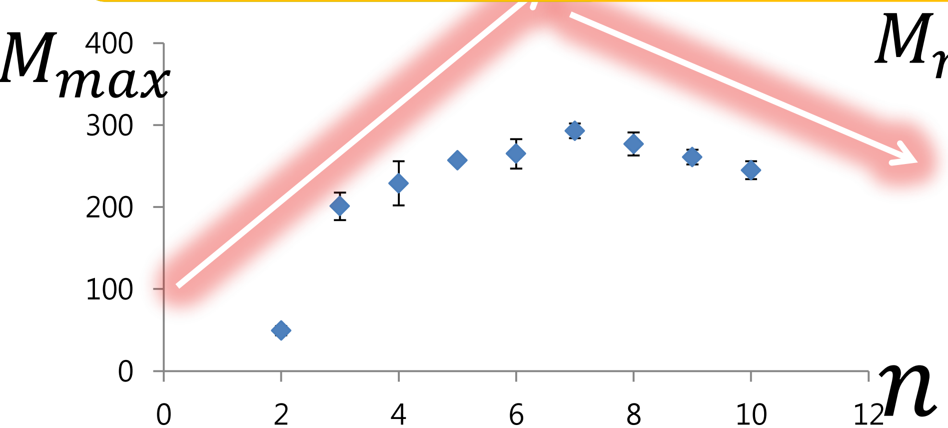
Reality(Bending+Necking+Sliding)

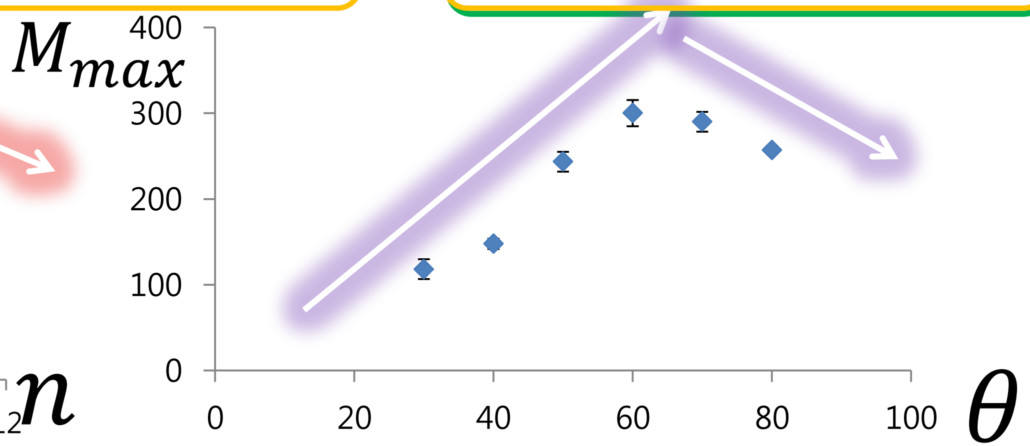




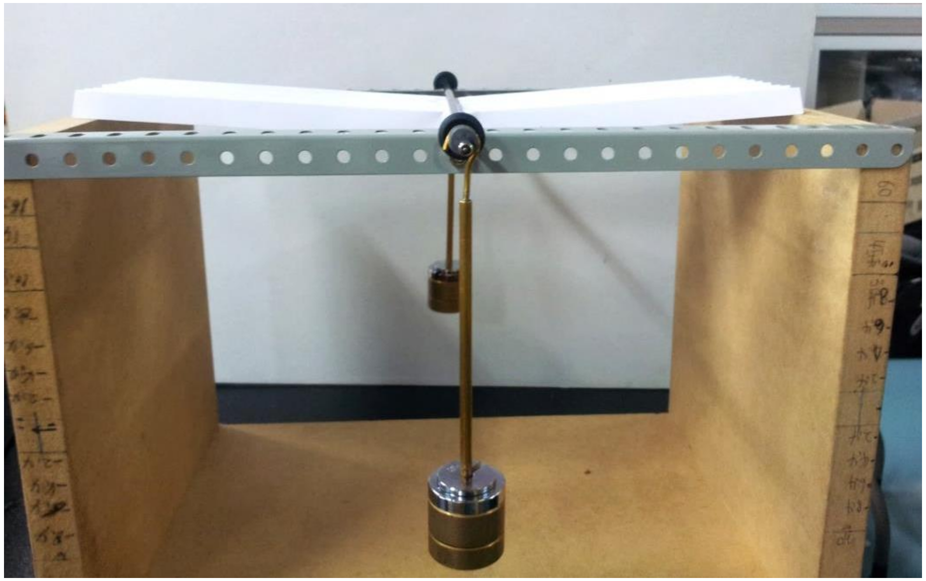
Parameters vs Strength







Optimization



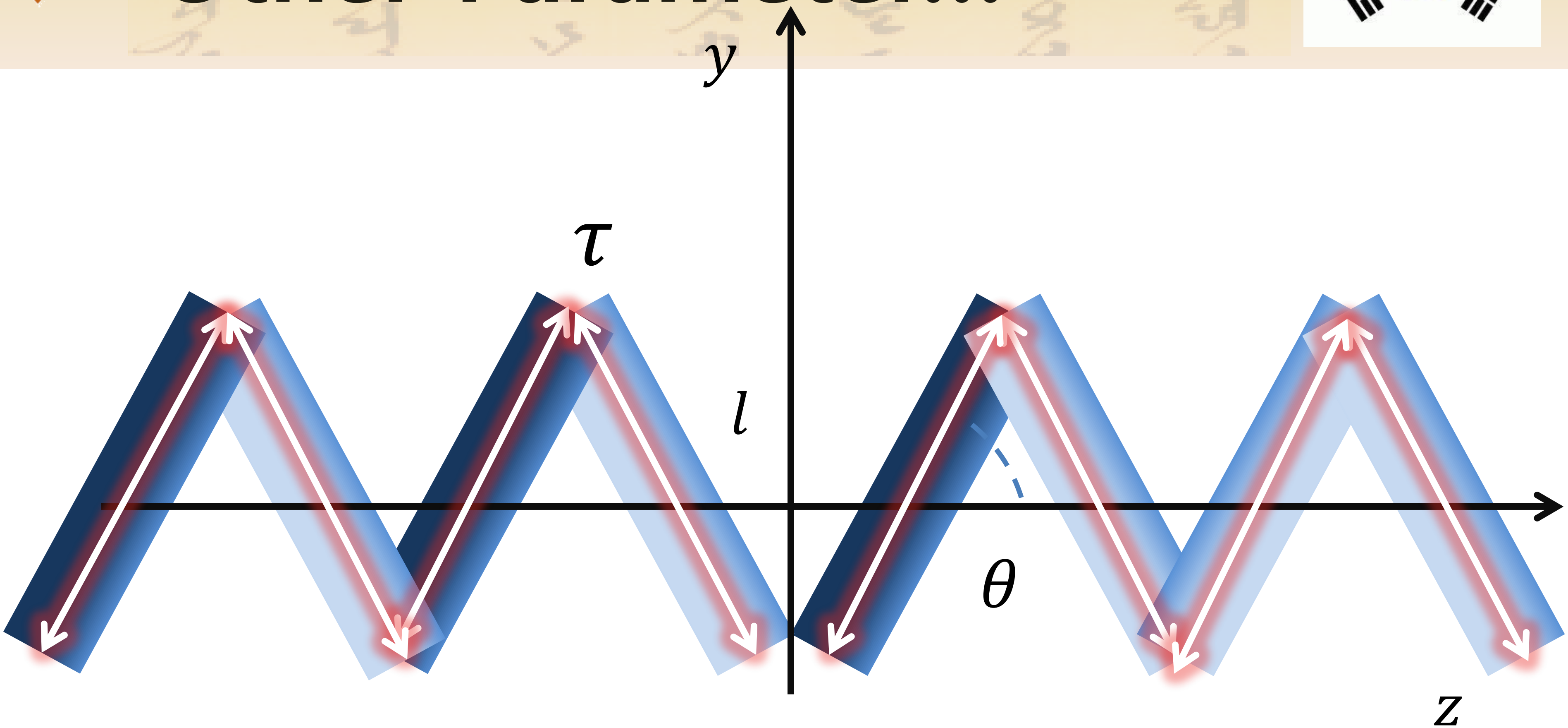
n=8, θ=60°

M_{max} =306.9g(F_{max} =3.01N)

n=7

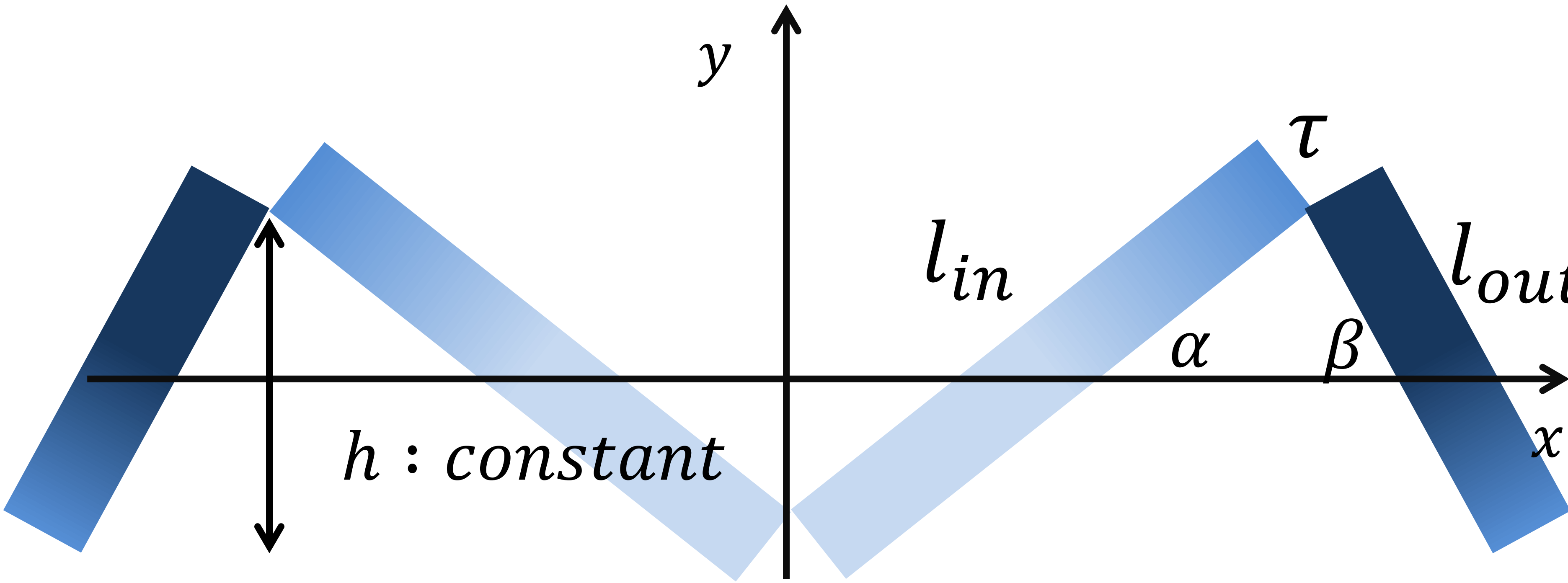
$M_{max.tube} = 296.2$g

Other Parameter...



How About Length Ratio?

Changing the Length Ratio



$$I_x = \frac{\tau l_{in}^3}{12}(1+\cos 2\alpha) + \frac{\tau l_{out}^3}{12}(1+\cos 2\beta)$$

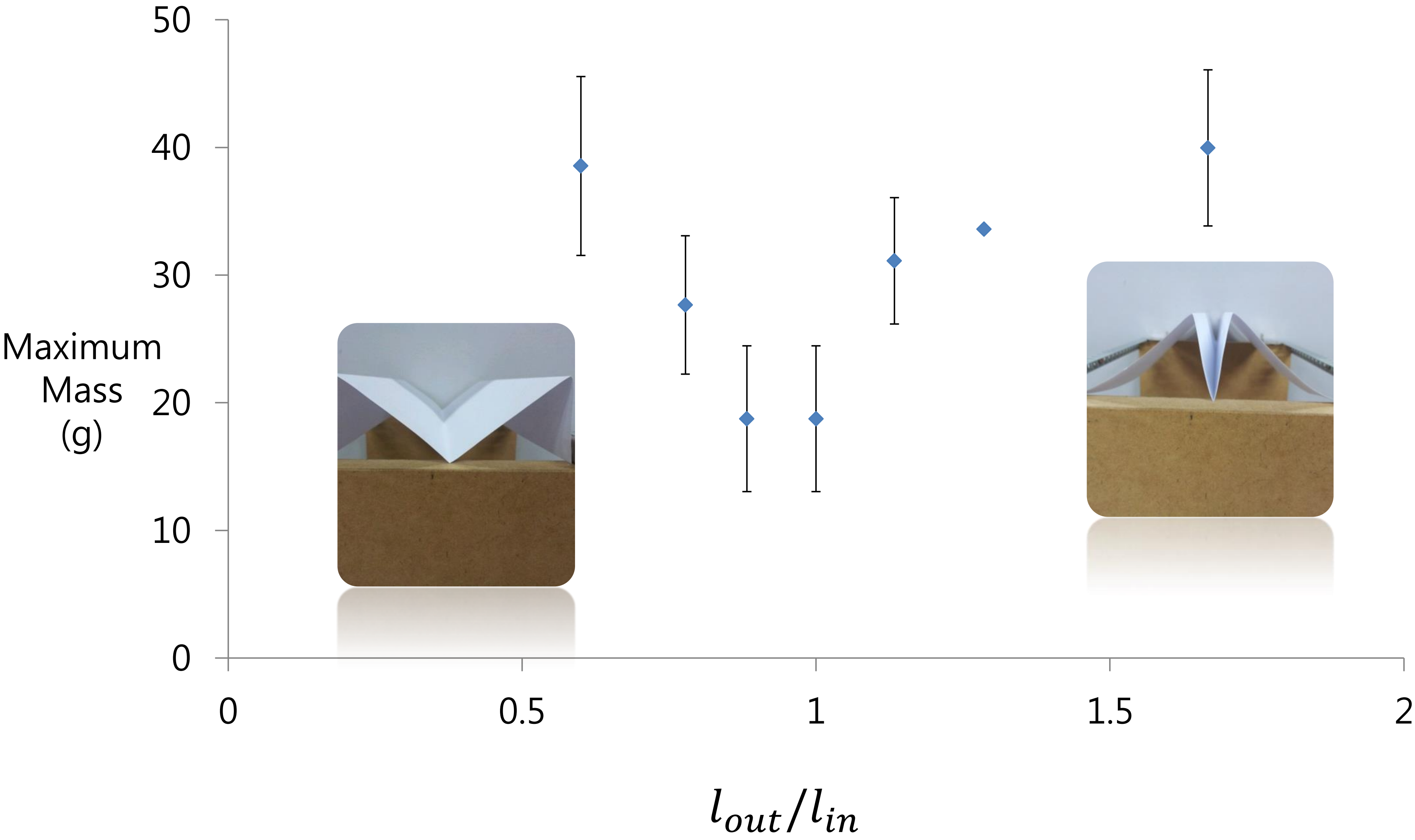
$$F_{max} = \frac{4\sigma_{max} I}{Ly}$$

$$l_{in} + l_{out} = \frac{D}{2}$$

$$y = \frac{h}{2}$$

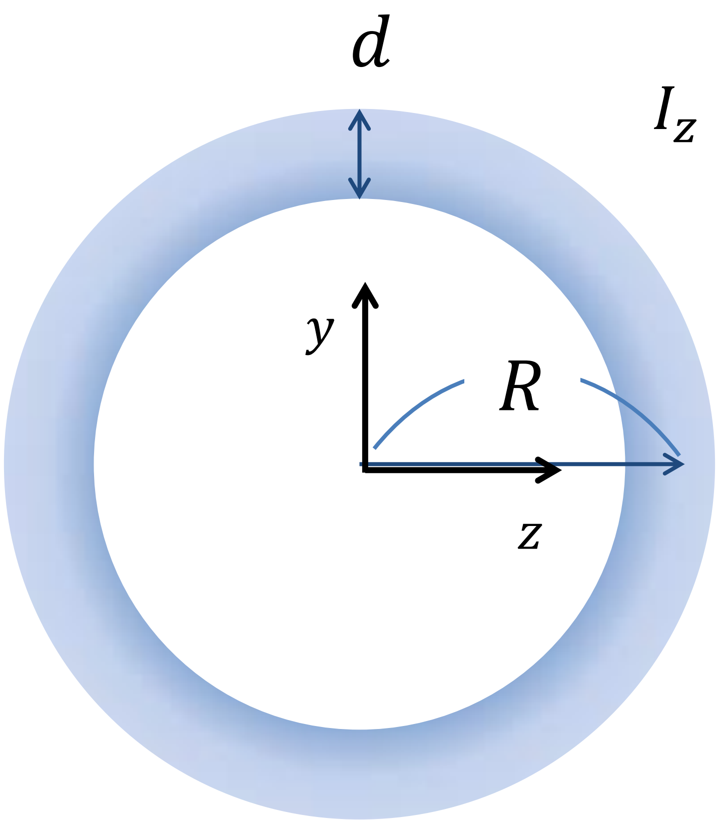
If $\alpha = \beta$ $\rightarrow I_x$, F_{max} minimized

Length Ratio(n=2, h=39mm)



Thank You

Second Moment of Area of a Tube



n- layers tube

$d \ll R$

$$I_z = \frac{\pi}{4}\left(\left(R+\frac{d}{2}\right)^4 - \left(R-\frac{d}{2}\right)^4\right) \approx \pi R^3 d$$

$$d = n\tau \qquad R = \frac{D}{2\pi n}$$

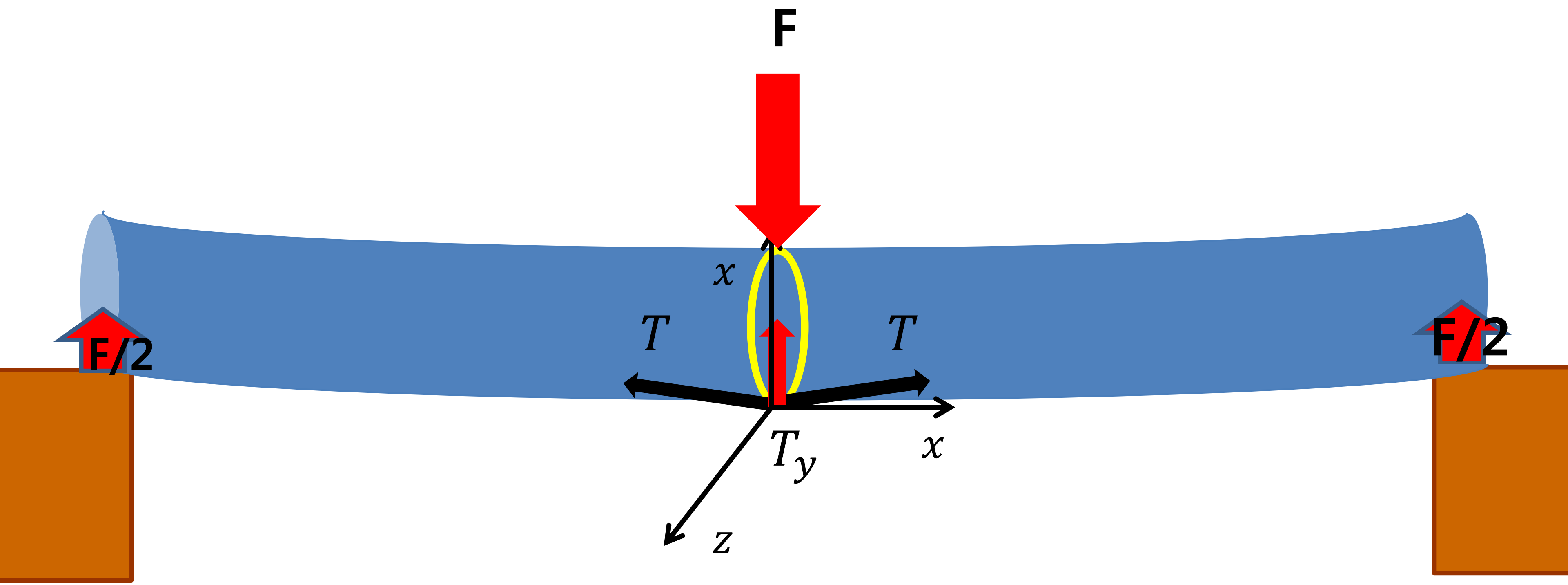
$$\boldsymbol{I_z = \frac{D^3\tau}{8\pi^2 n^2}}$$

*τ : Thickness of an A4 sheet

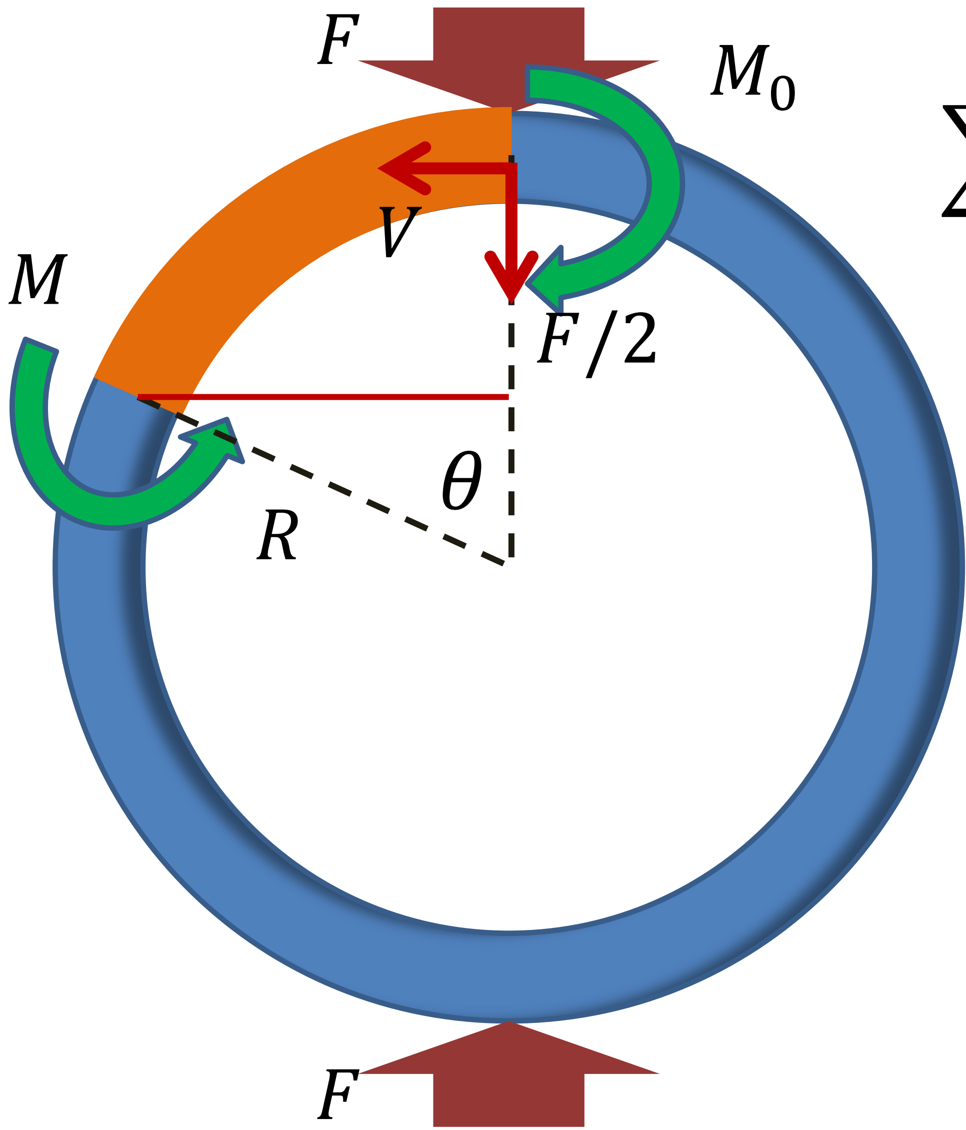
*D : Width of an A4 sheet

*A : Cross-section Area

Local Deformation in Tube Style



Bending Moment of a Beam



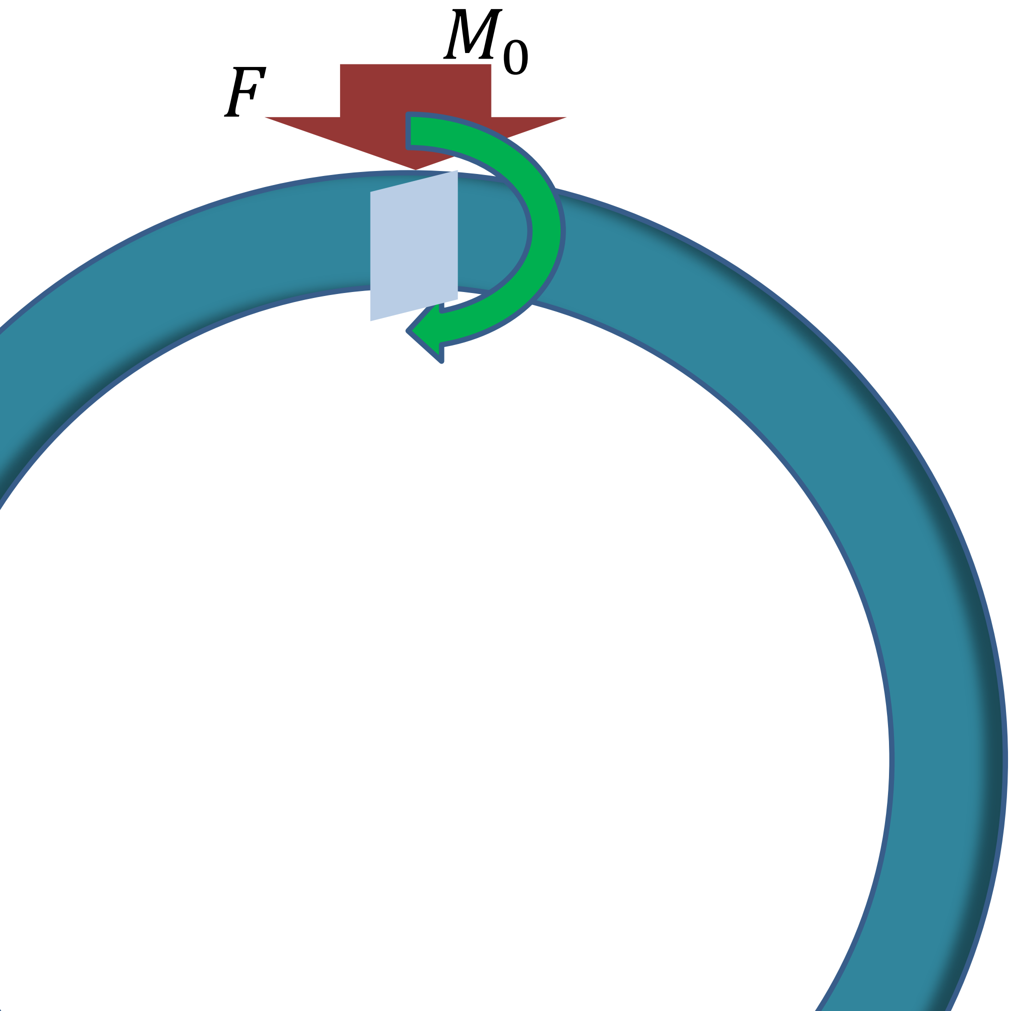
$$\sum M = M + M_0 - \frac{FR\sin\theta}{2} - VR(1 - \cos\theta) = 0$$

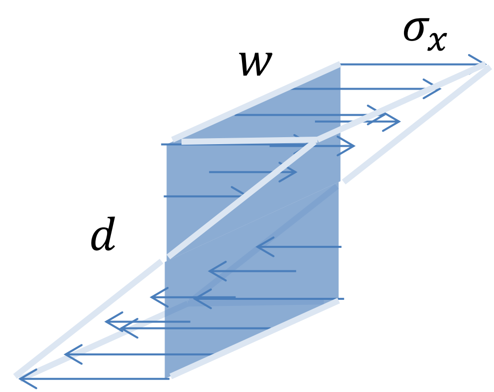
$$\delta_x = \frac{\partial U}{\partial V} = \frac{1}{EI}\int_0^{\pi} M\frac{\partial M}{\partial V}R\,d\theta$$

$V = 0$ $\delta_x = 0$

$$\boldsymbol{M_0 = \frac{FR}{\pi}}$$

Compressive Stress at the Top



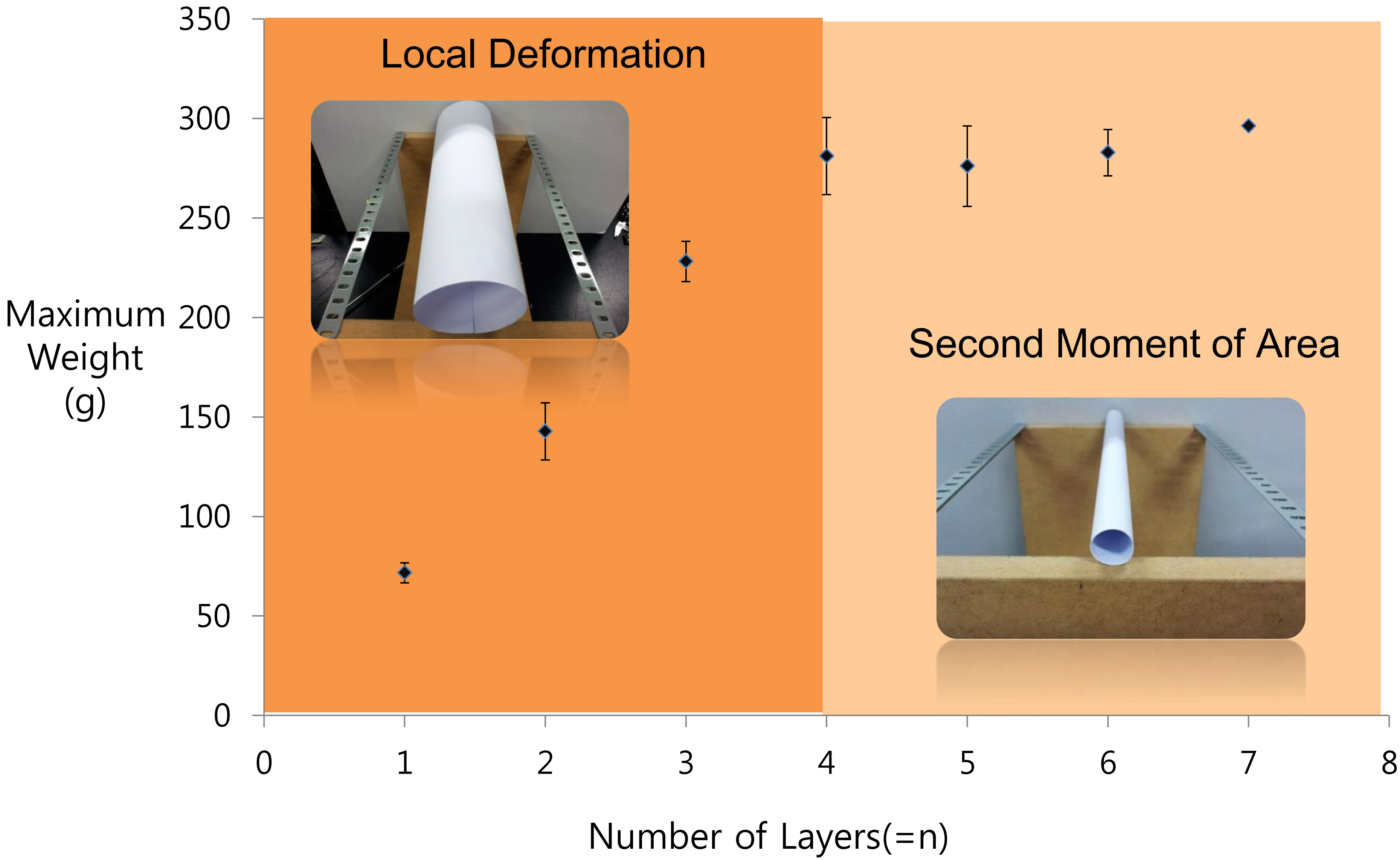


$$M_0 = \frac{\sigma_x I}{y} = \frac{\sigma_x w d^2}{6} = \frac{FR}{\pi}$$

$$F = \frac{\pi w d^2 \sigma_x}{6R} \leq \frac{\pi w d^2 \sigma_{max}}{6R}$$

$$\boldsymbol{F_{max} = \frac{\pi^2 w \tau^2 \sigma_{max}}{3D} n^2}$$

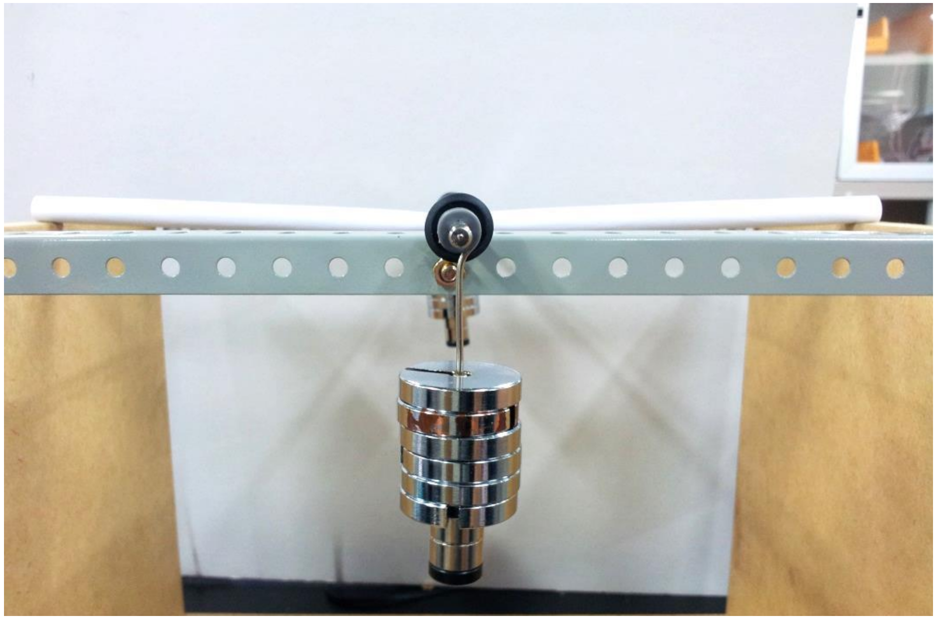
Number of Layers vs Strength



Further Investigation

1) Force distribution
2) Number of layers of the truss

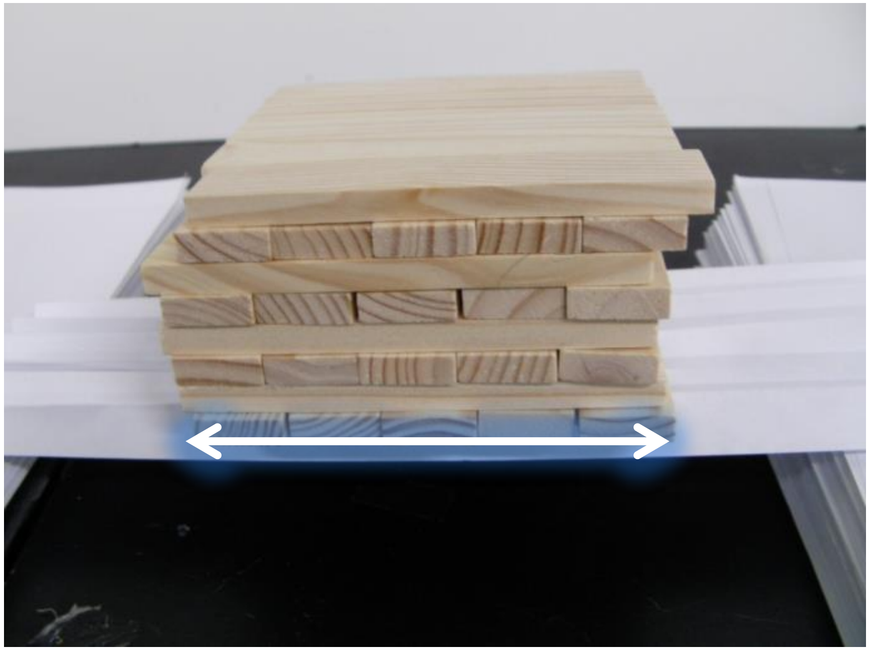
Optimization



n=7

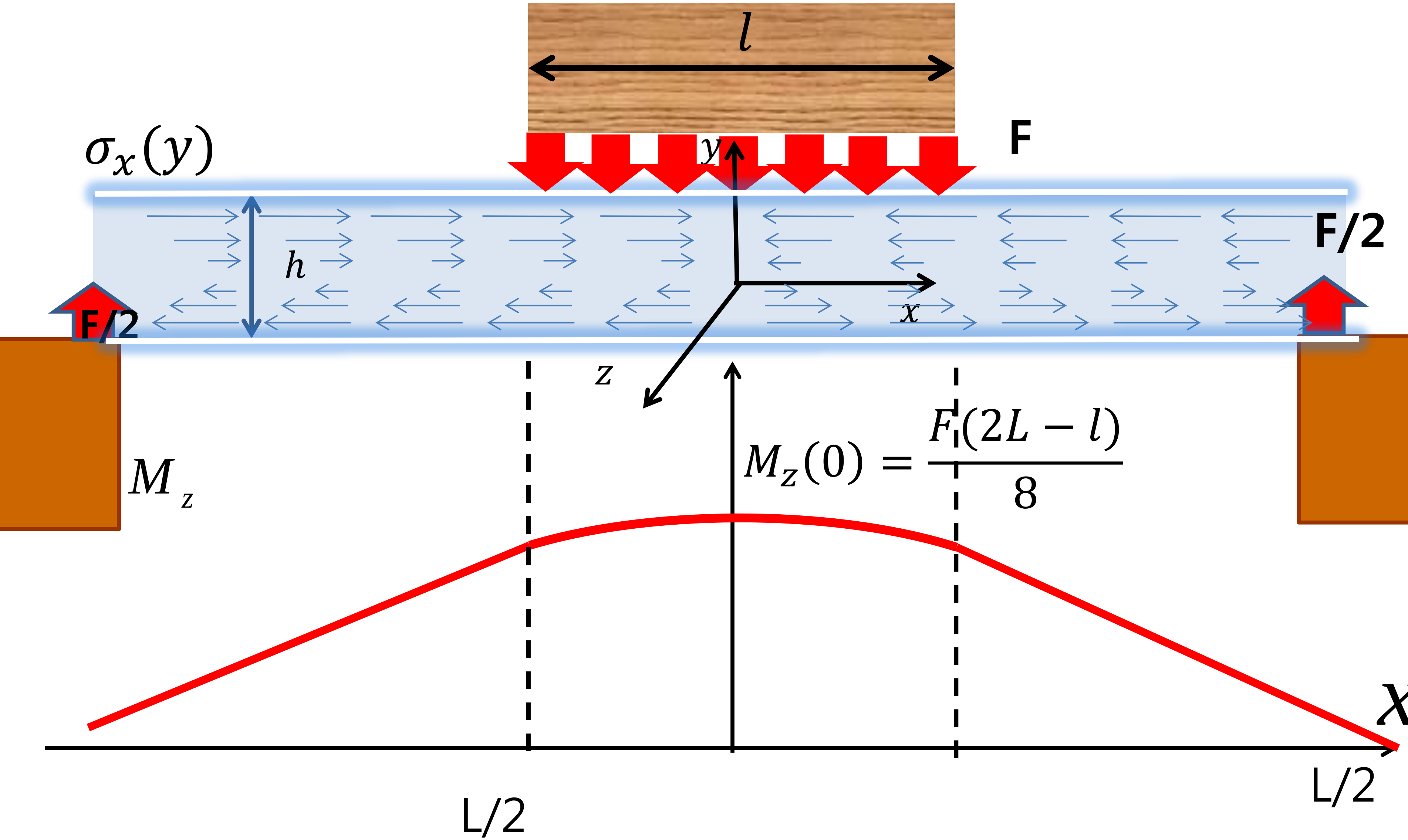
$M_{max.tube} = 296.2g \quad < M_{max.truss} = 306.9g$

Force distributed

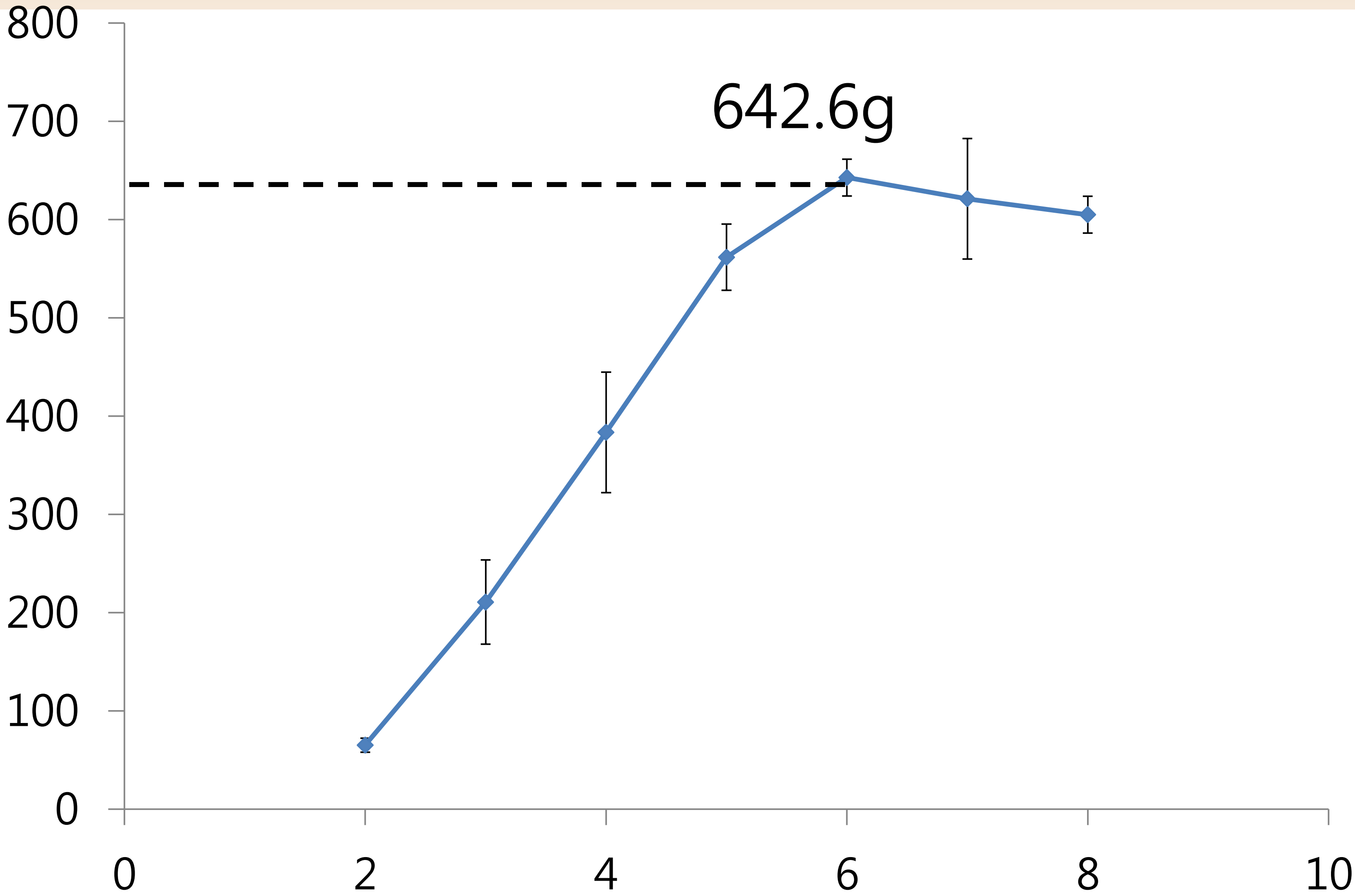


150mm

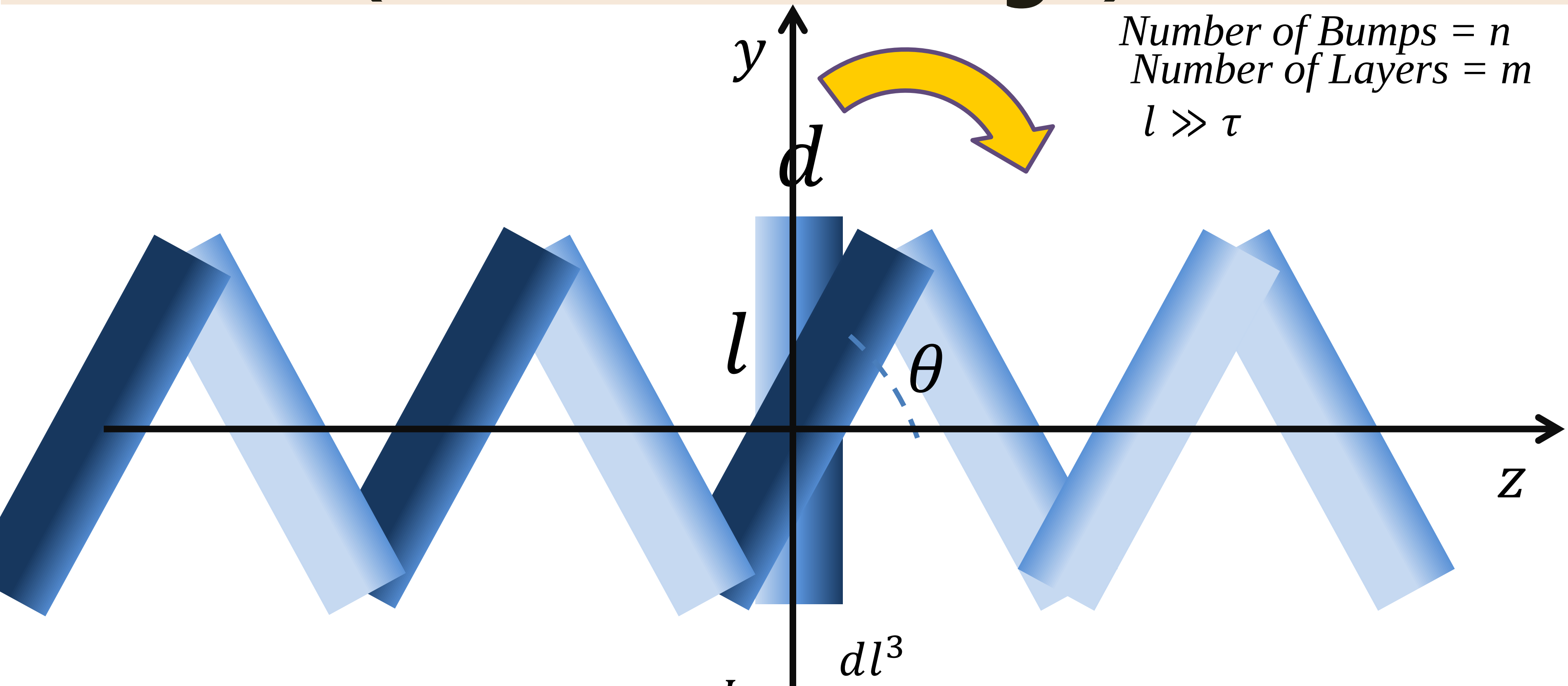
Distributed Mass Case



Distributed Mass Case



Second Moment of Area(thickness change)



$Number\ of\ Bumps = n$

$Number\ of\ Layers = m$

$l \gg \tau$

$$I_{z0} = \frac{dl^3}{12}$$

$$I_{z\theta} \approx \frac{ndl^3}{12}(1 - \cos 2\theta)$$

$$l = \frac{D}{2nm}$$

$$d = m\tau$$

$$\boldsymbol{I_z = \frac{D^3\tau}{96(nm)^2}(1 - \cos 2\theta)}$$

Local Deformation(thickness change)

$$M(0) = \frac{FL}{4} = \frac{\sigma_{top} I}{y} \qquad \sigma_{top} \geq \sigma_{max}$$

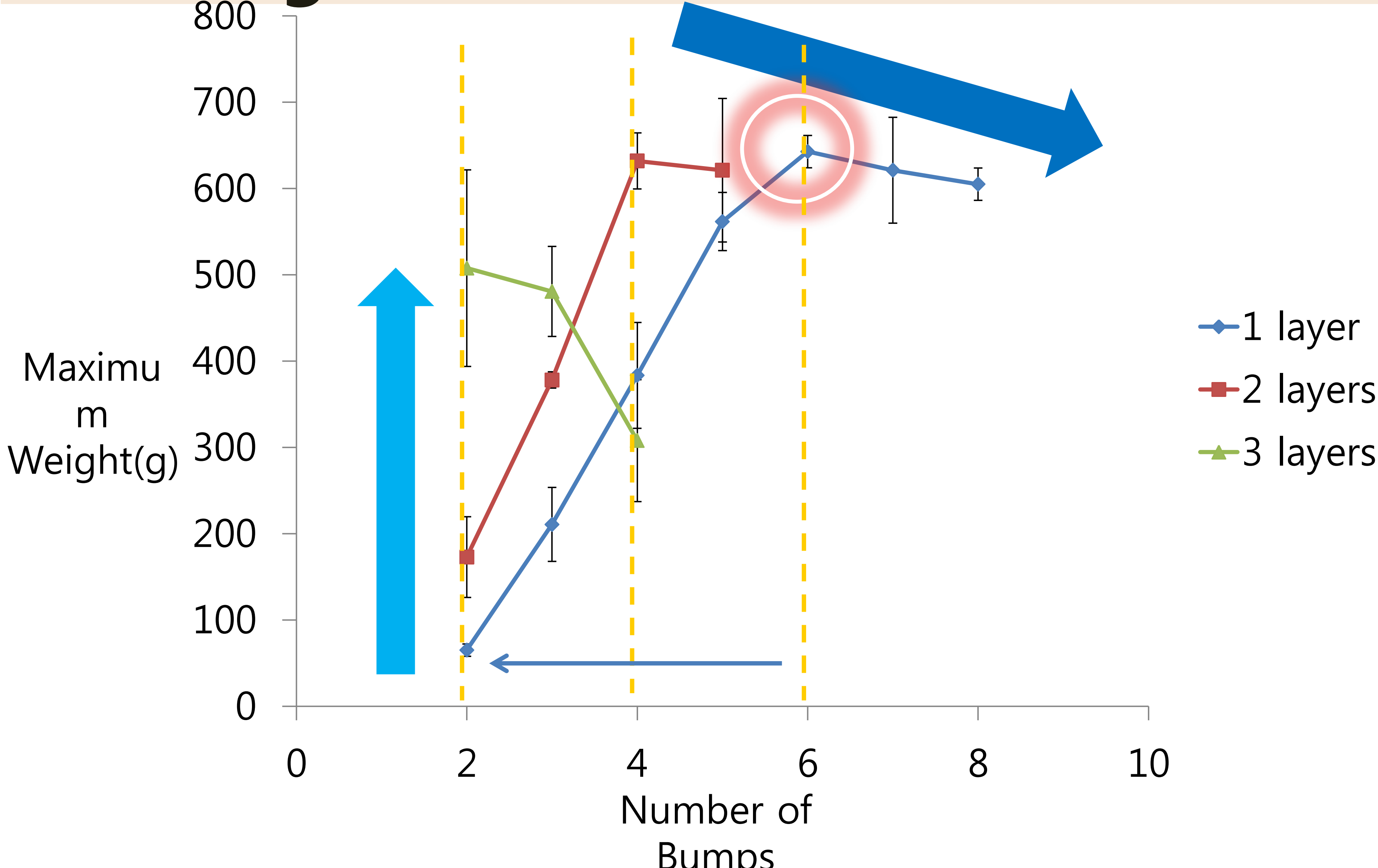
$$F_{max} = \frac{4\sigma_{max} I}{Ly}$$

$$y = \frac{h}{2} = \frac{D \sin\theta}{2nm} \qquad I_z = \frac{D^3 \tau}{96(nm)^2}(1 - \cos 2\theta)$$

$$F_{max} = \frac{\sigma_{max} D^2 \tau \sin\theta}{6nmL}$$

→ Thickness change does not make the bridge stronger!

Thickness vs Maximum Weight

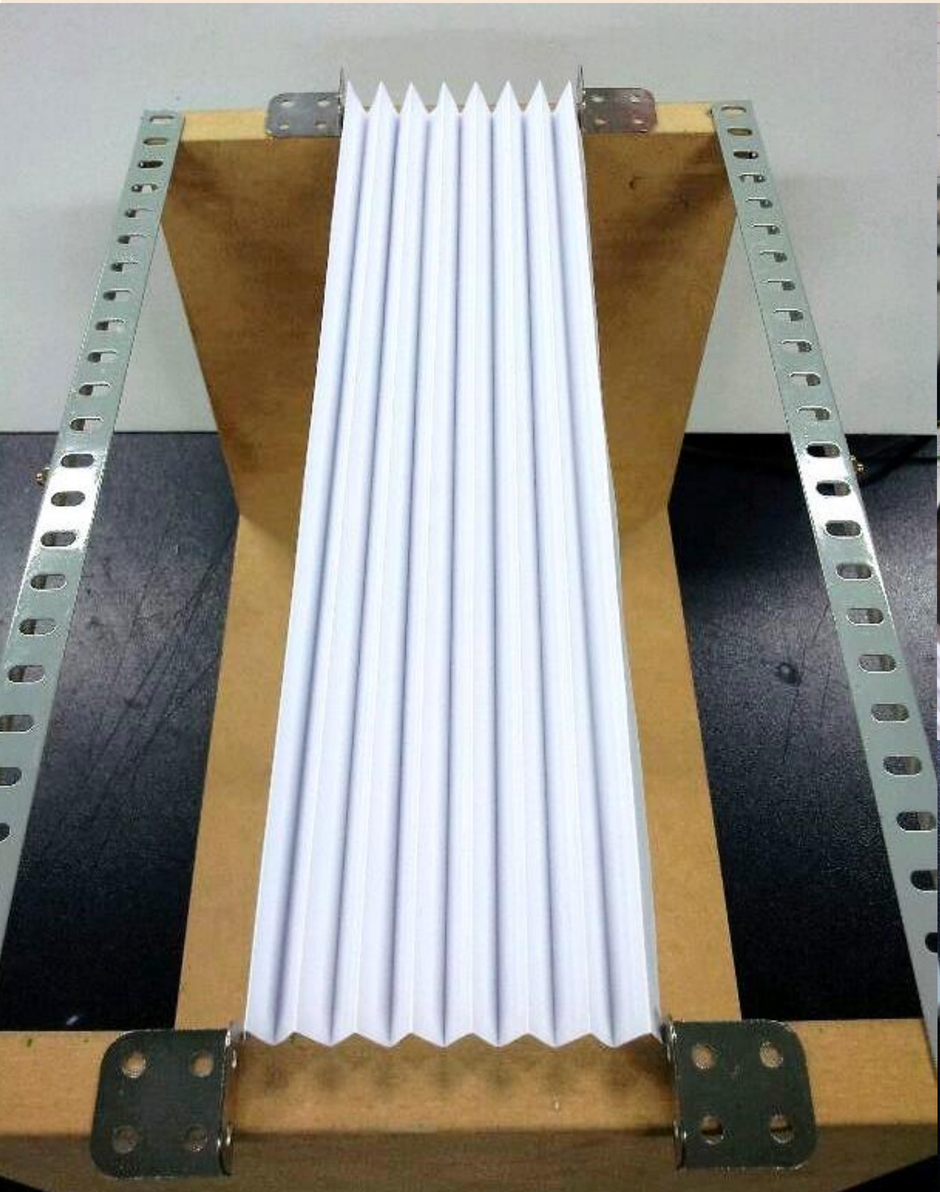


Humidity Control

Humidity Range
74%~80%

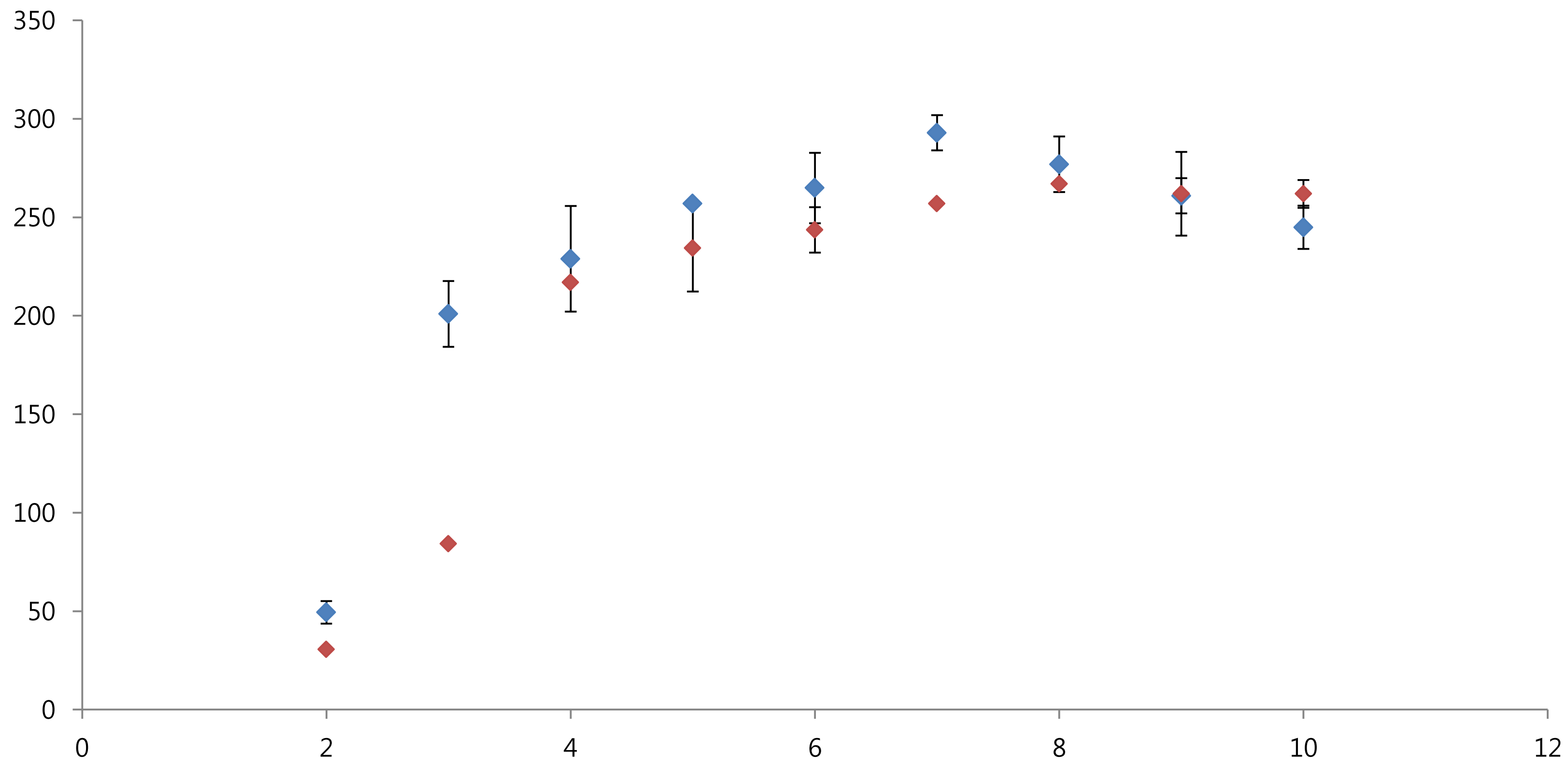
→ Hydrogen bond btw cellulose change
→ Hard to quantify

W/o Sliding Effect



Fixed ends – Prevents sliding to the side

Eliminating the Sliding



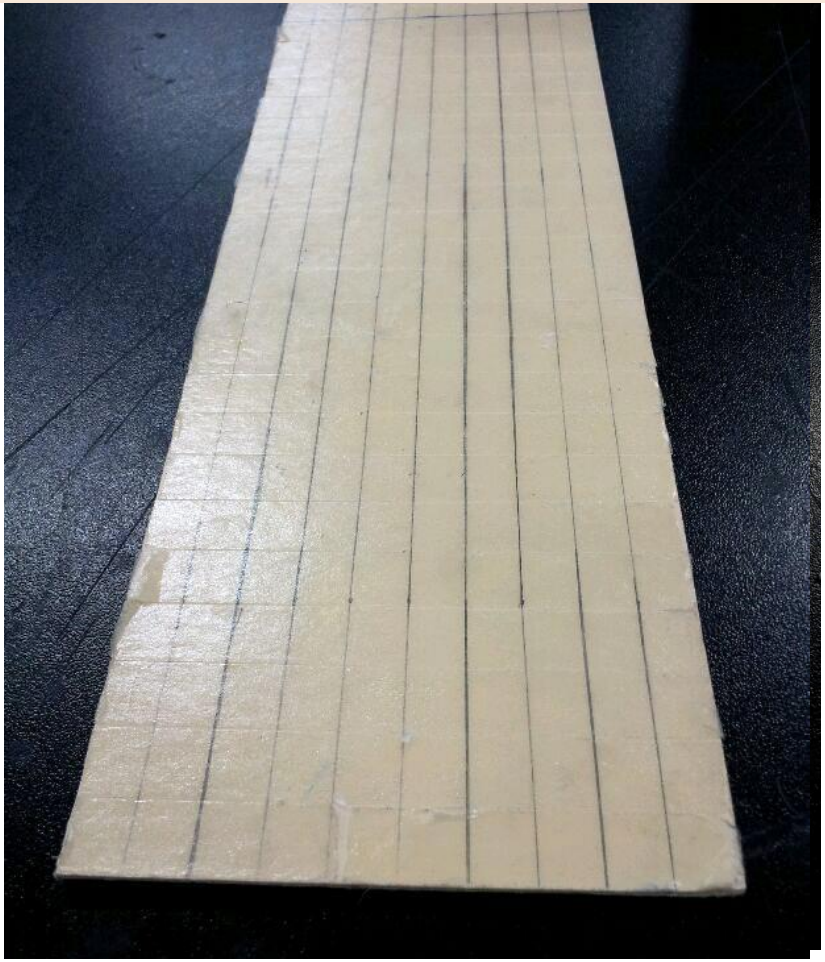
Experimental Setup

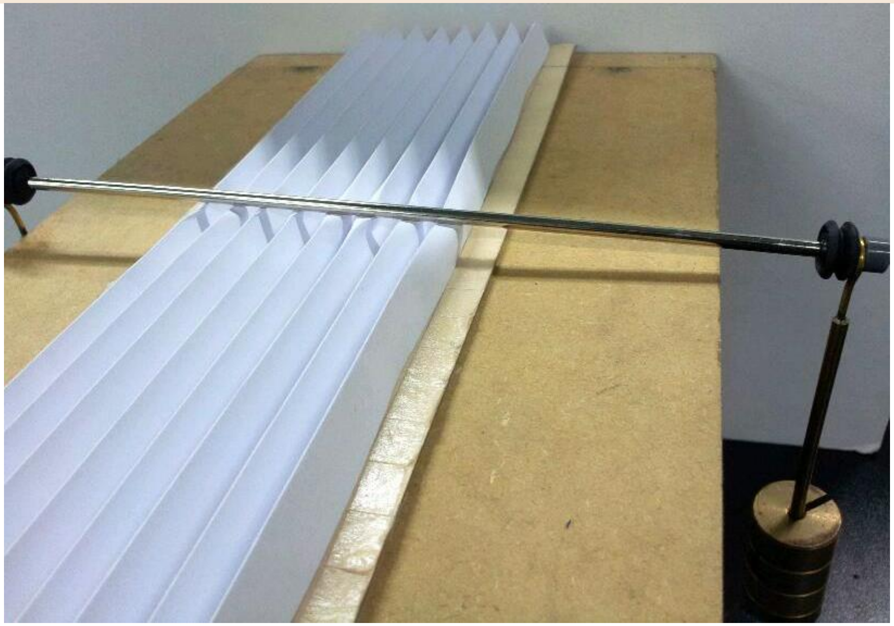
질량, 두께의 A4용지 사용, small amount of glue 정의는 반론 슬라이드, gap은 나무로 됨, 하중을 올리는 방법, collapse의 정의(하중을 두는 막대기 한 쪽 또는 양 쪽이 바닥 높이와 같아질 때 = 종이가 약하기 때문에 영구 변형되는 것과 막대기가 바닥으로 닿는 것의 차이가 크지 않음, 거의 같은 무게에서 그렇게 됨.)

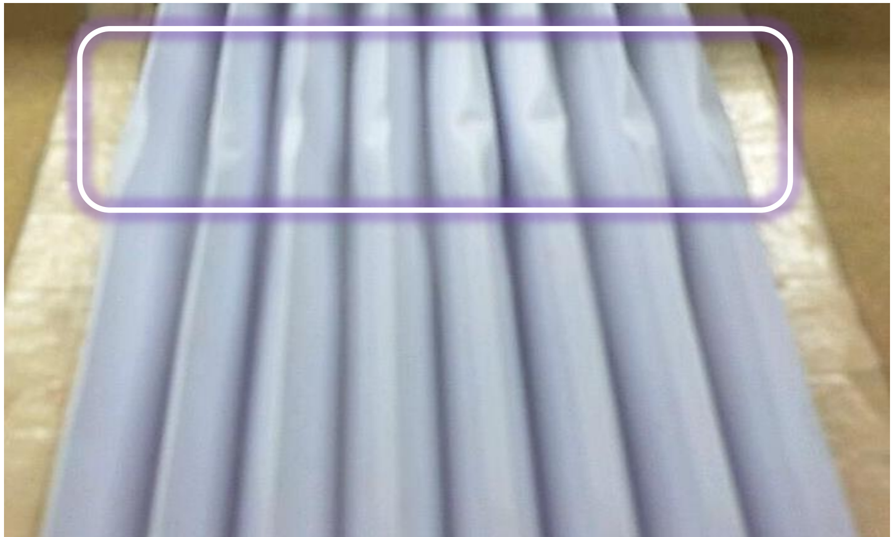
Friction

- Bump의 개수에 따라 normal force는 나누어져서 줄지만 horizontal force도 그만큼 작아짐→데이터 제시(N=1~5결과)
- 길이의 Imbalance가 생기면 길어서 contact angle이 작은 쪽이 더 작은 y방향 힘을 받으니 더 미끄러지기 쉽다.(unstable한 local minimum임) → 데이터 제시 ((ex)N=3)

Collapse Due to Necking

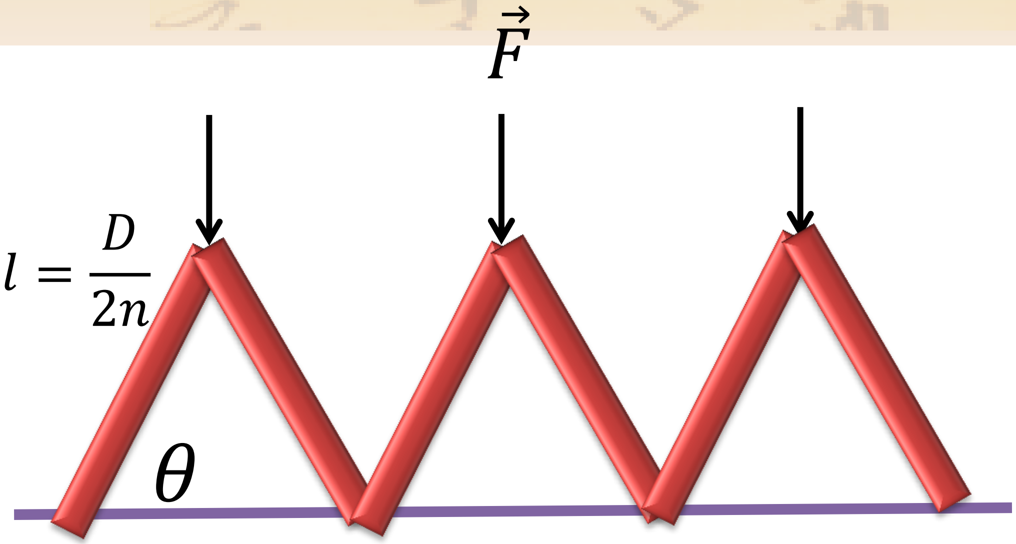


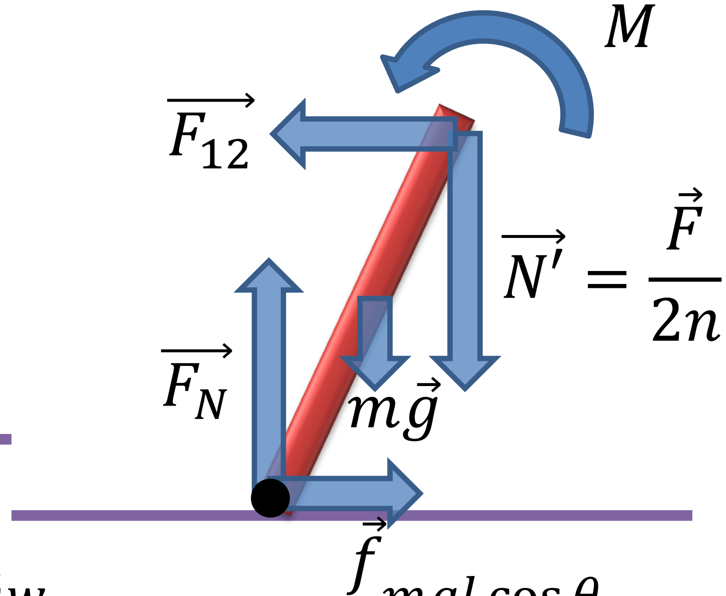




Attached to the ground
→ eliminate bending & sliding

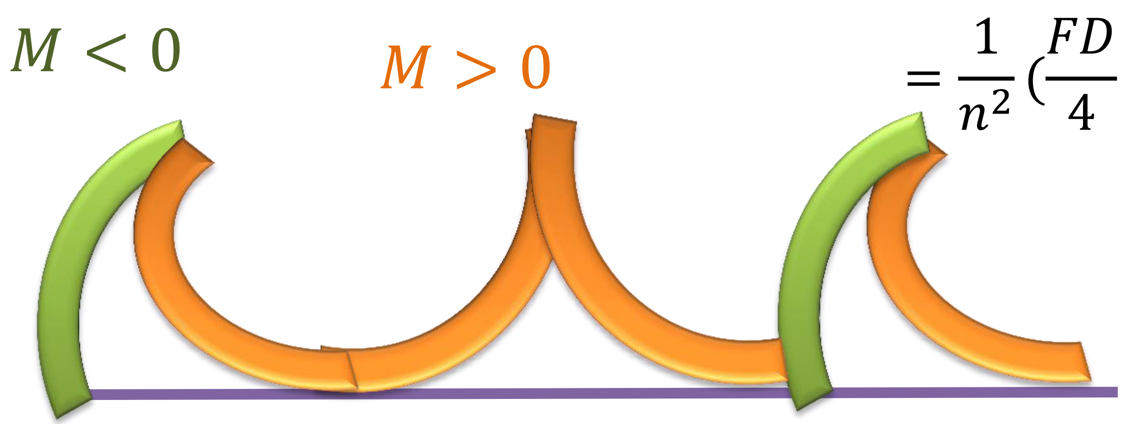
Second Scenario-Necking



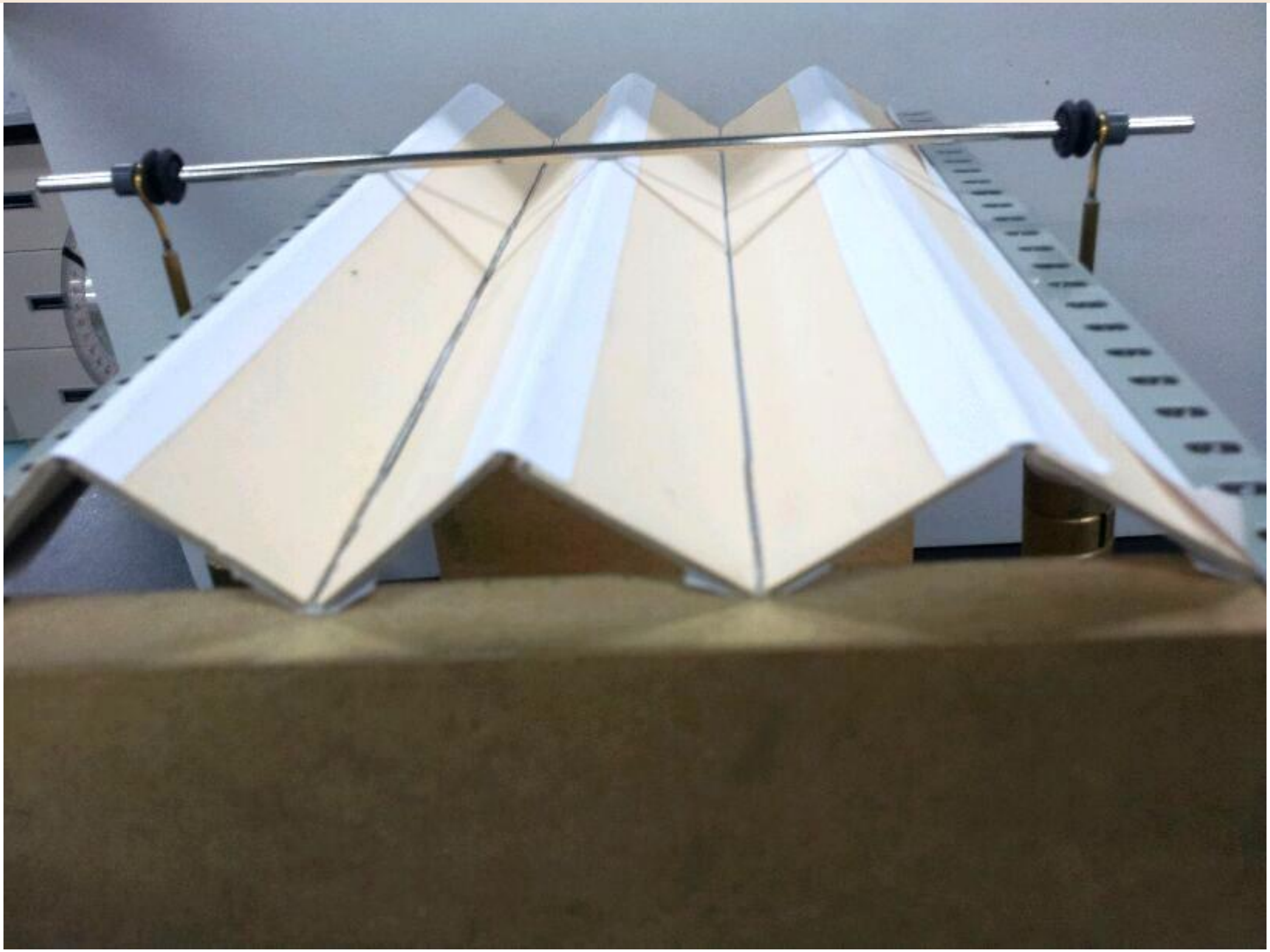


$$M = -EI\frac{d^2w}{dx^2} = N'l\cos\theta + \frac{mgl\cos\theta}{2} - F_{12}l\sin\theta$$

$$= \frac{1}{n^2}\left(\frac{FD}{4}\cos\theta + \frac{MgD\cos\theta}{8}\right) - \frac{fD}{2n}\sin\theta$$



Collapse Due to Sliding

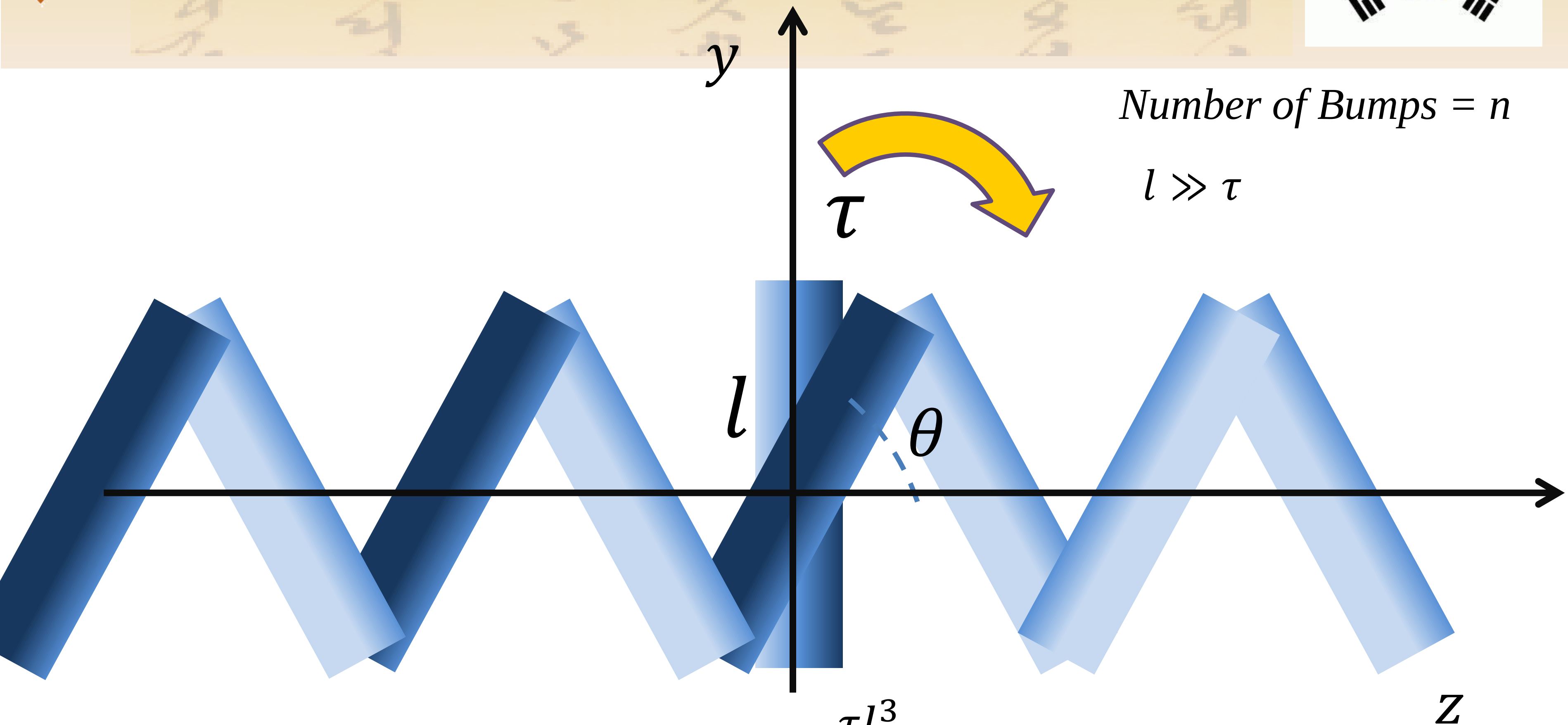


Thick paper bridge → eliminate necking and bending

Second Moment of Area

$Number\ of\ Bumps = n$

$l \gg \tau$



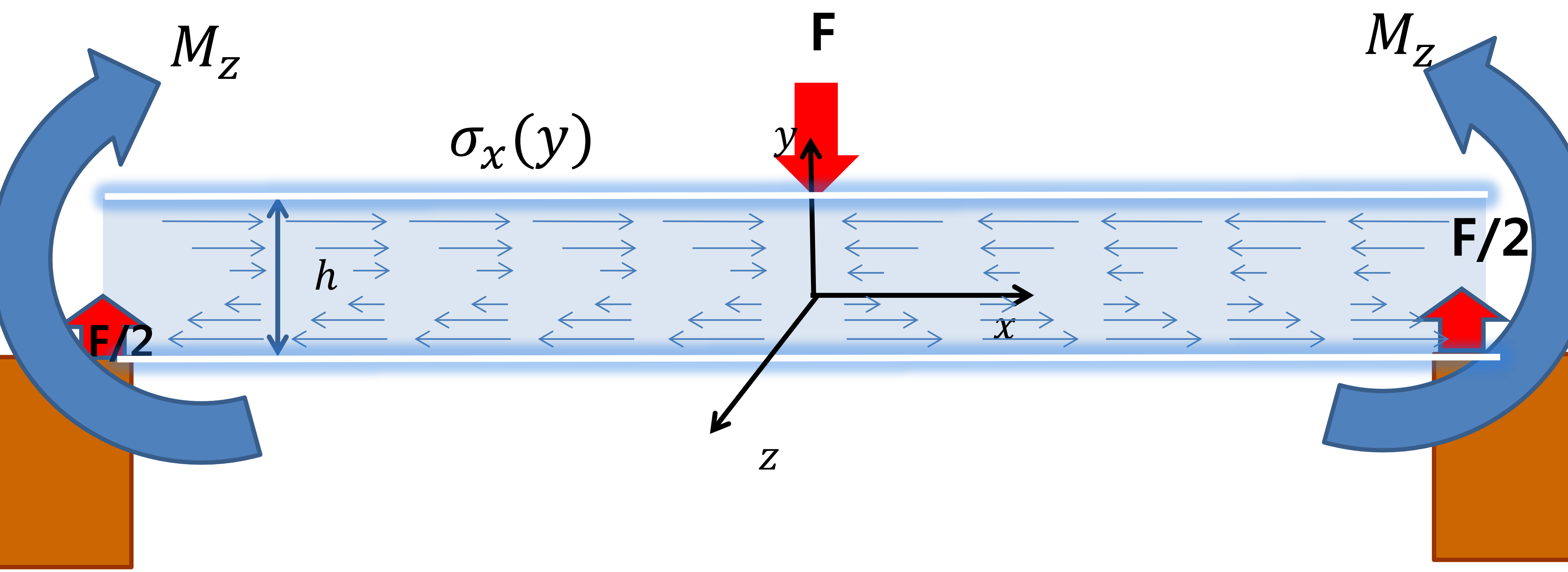
$$I_{z0} = \frac{\tau l^3}{12}$$

$$I_{z\theta} \approx \frac{n\tau l^3}{12}(1 - \cos 2\theta)$$

$$l = \frac{D}{2n}$$

$$\boldsymbol{I_z = \frac{D^3 \tau}{96 n^2}(1 - \cos 2\theta)}$$

Ultimate Compressive Stress

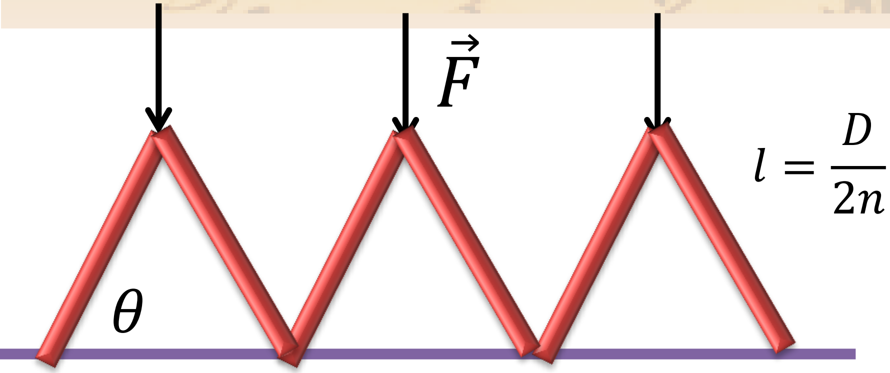


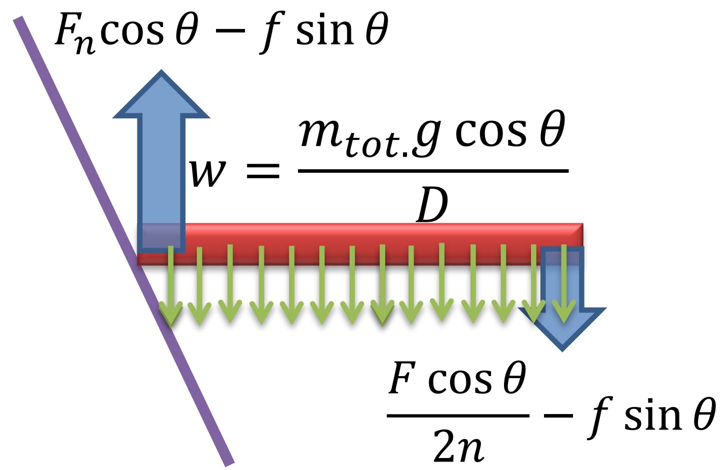
Cellulose : ultimate compressive stress≤ultimate tensile stress

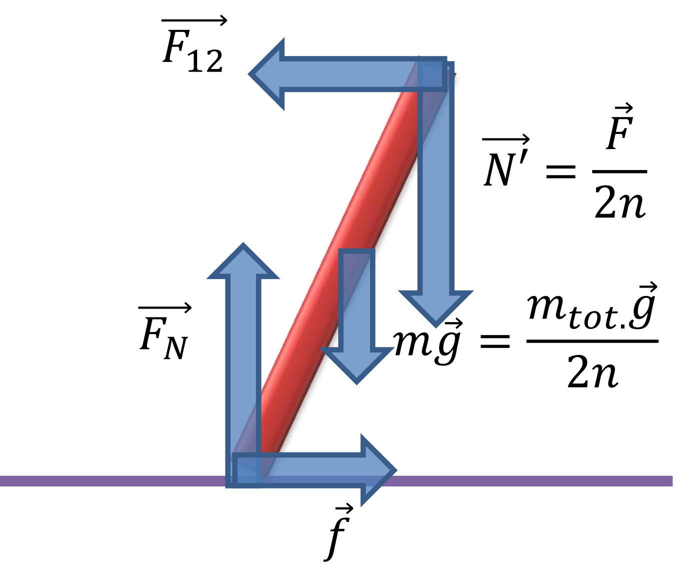
*Waterhouse, John F. "The ultimate strength of paper." (1984).

$\boldsymbol{\sigma_x(h) \geq \sigma_{max}}$ → **Local Deformation**

Third Scenario-Necking(Before ultimate compressive limit)







Force Equilibrium

$$F_{12} = f$$

$$F_N = \frac{1}{2n}(F + m_{tot}.g)$$

Shear Force and Bending Moment

$$M(x) = \left(\frac{F + m_{tot}.g}{2n}\cos\theta - f \sin\theta\right)x - \frac{m_{tot}.g \cos\theta}{2D}x^2$$

$$M(x) = -EI\frac{d^2w}{dx^2}$$

$$\boldsymbol{abs\left(\frac{d^2w}{dx^2}\right) = \frac{12D}{E\tau^3 Ln}\left(F - f \sin\theta + \frac{m_{tot}.g \cos\theta}{4n}\right)}$$

Shift of the Load