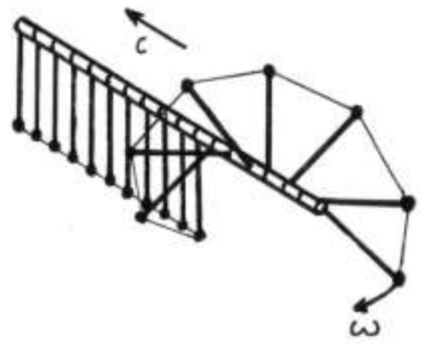
4.

Soliton

Marco Bodnár

Task

A chain of similar pendula is mounted equidistantly along a horizontal axis, with adjacent pendula being connected with light strings. Each pendulum can rotate about the axis but can not move sideways (see figure).



- Investigate the propagation of a deflection along such a chain.

- What is the speed for a solitary wave, when each pendulum undergoes an entire 360º revolution?

SLOVAKIA
IYPT '13

Terms & Definitions

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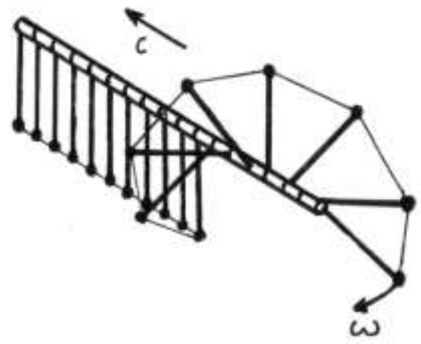
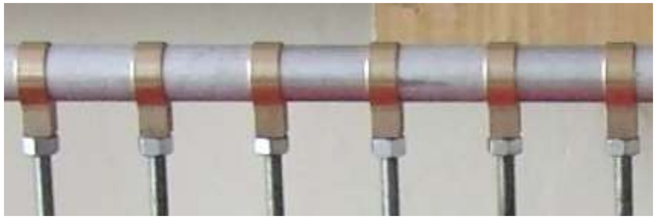


Equidistantly mounted pendula

⇓

All strings are straightened equally



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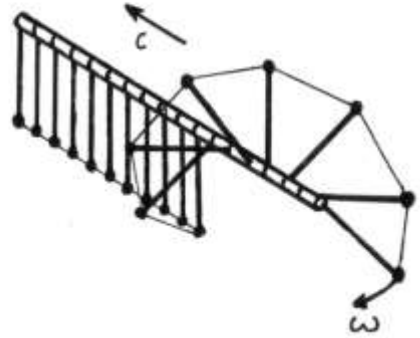

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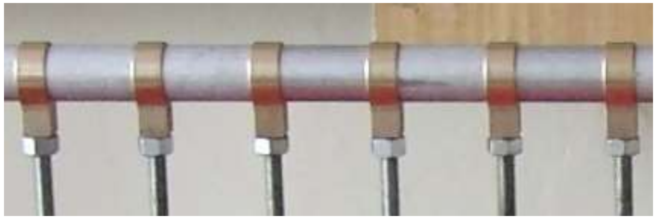

Using rubber as string

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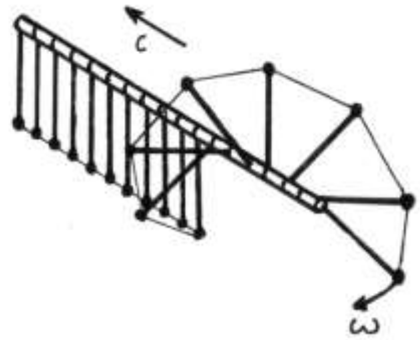

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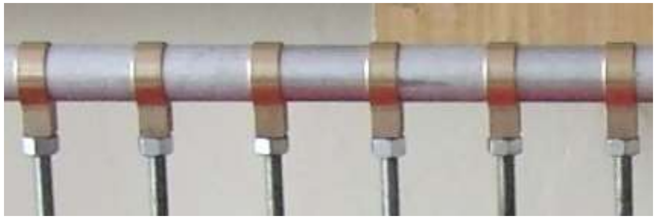
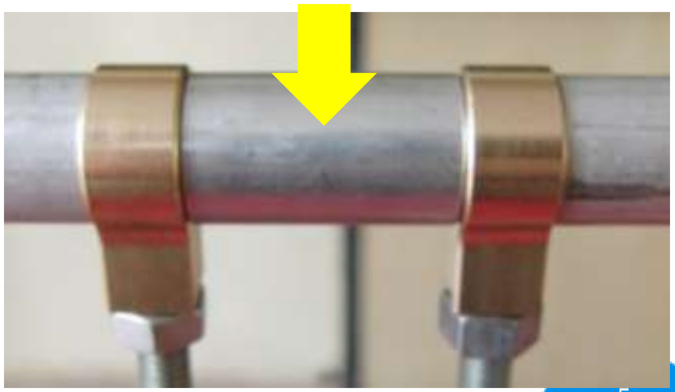


Using rubber as string

Small tube between pendula



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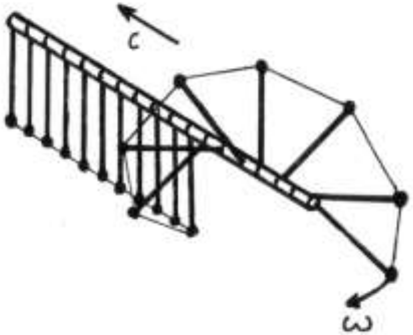
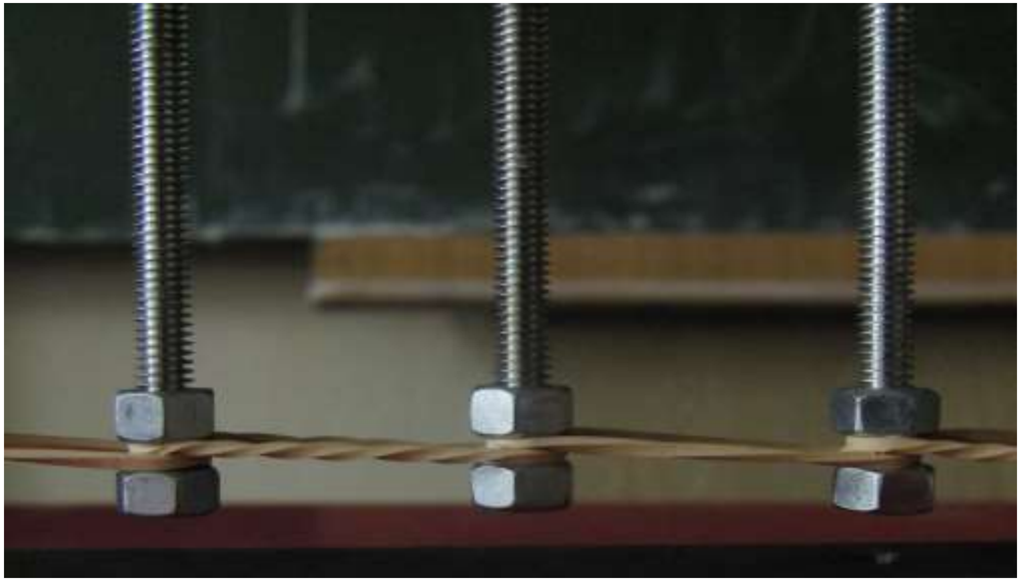
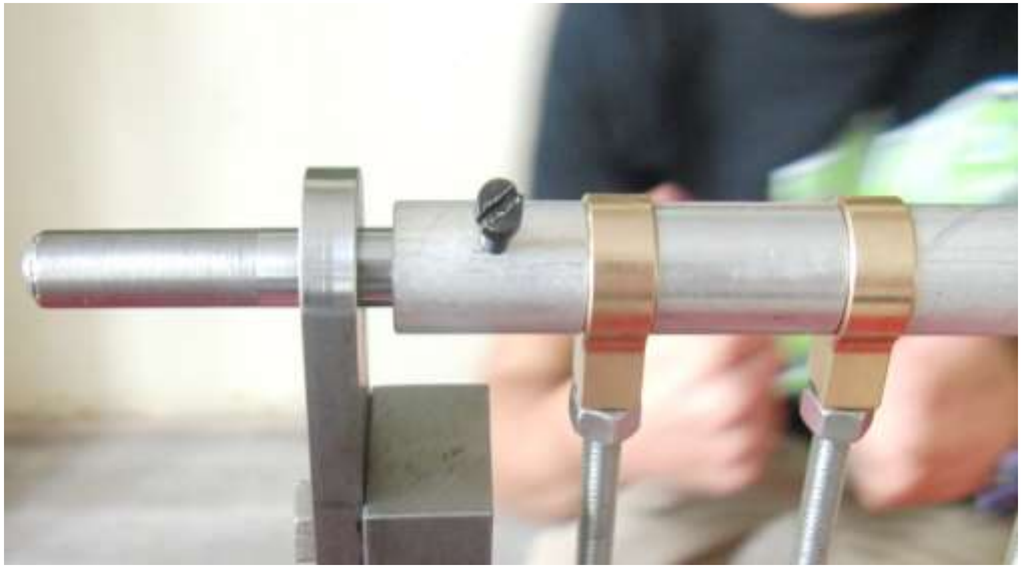


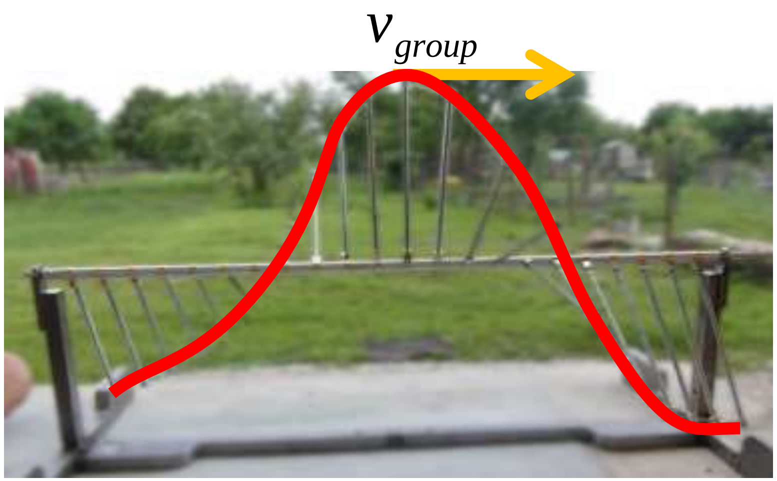
Chain of pendula





Which speed?

Group velocity



$$v_{group} \equiv \frac{\partial \omega}{\partial k}$$

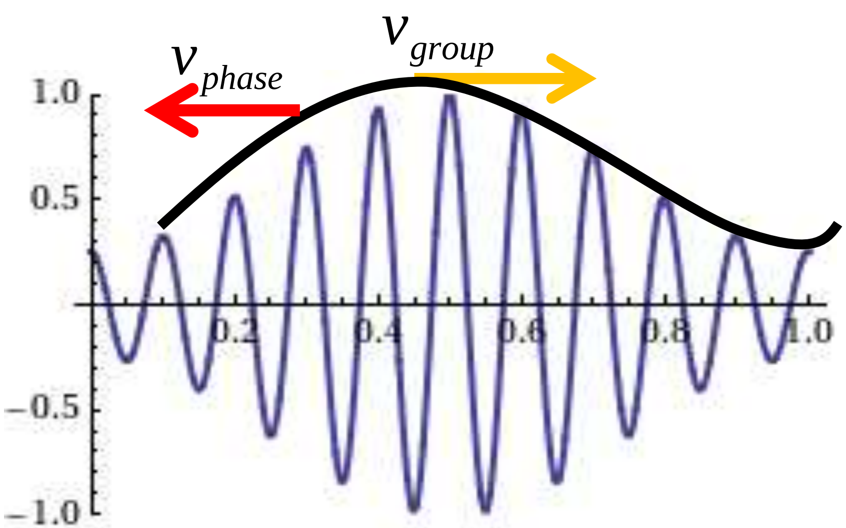
ω Angular frequency

Wave number

$$k = \frac{2\pi}{\lambda}$$

- Velocity of overall shape
- Velocity of the information

Phase velocity



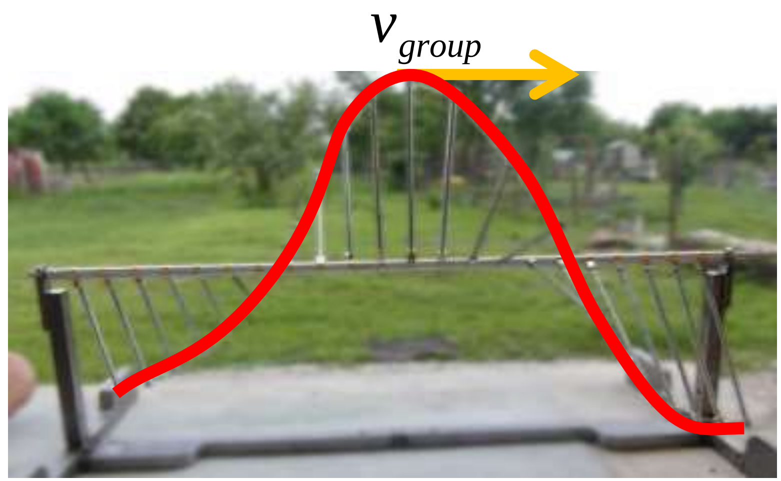
$$v_{phase} \equiv \frac{\omega}{k}$$

- Velocity of phase
- Can be higher than velocity of information or even has different direction

Which speed?

Group velocity

- Velocity of overall shape
- Velocity of the information



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Dispersion

- Property of a given system – Group velocity is wavelength depended

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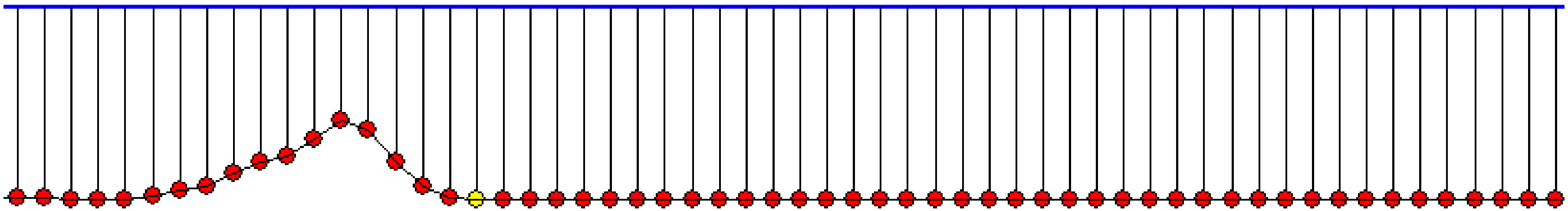

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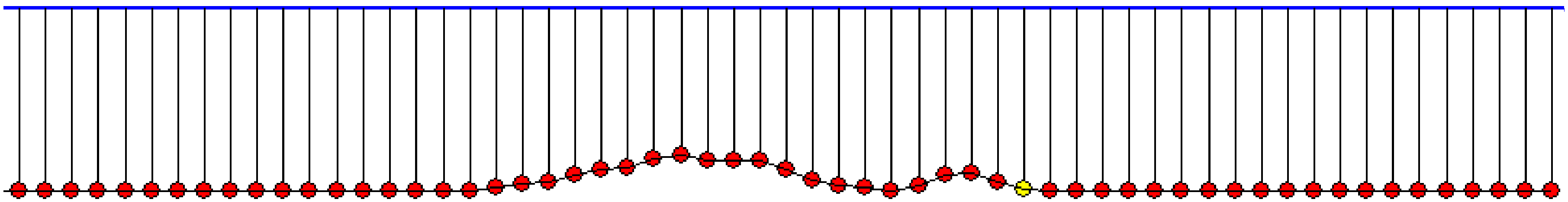

Dispersion

- Property of a given system – Group velocity is wavelength depended

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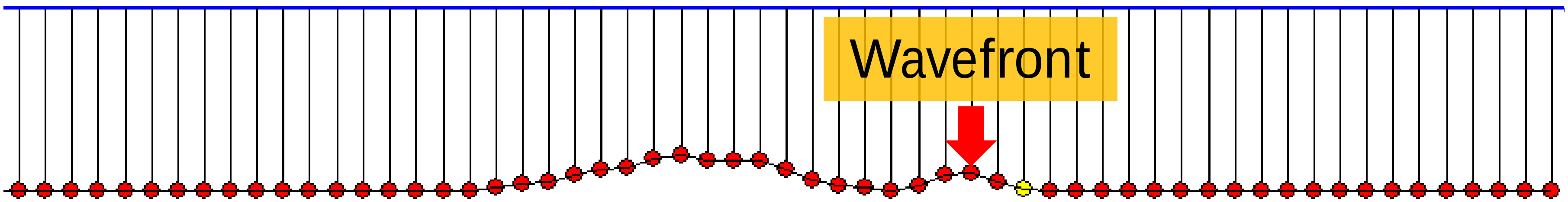
Breakdown of wave

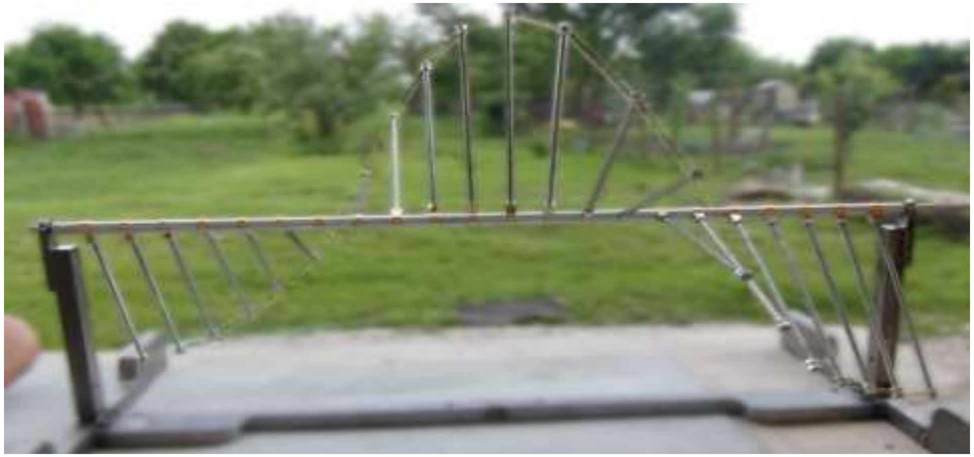


What is solitary wave?

- What is the speed for a solitary wave, when each pendulum undergoes an entire 360º revolution?

Photo from Physics of solitons. M. Peyrard, T. Dauxios, *Cambridge University Press (2010), ISBN 9780521143608*

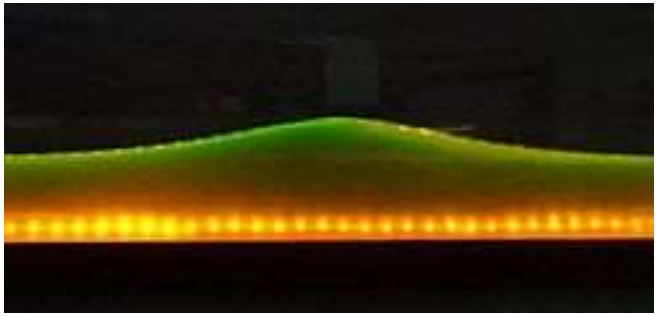
Place of first observed solitary wave (soliton) (Union Canal, Scotland)



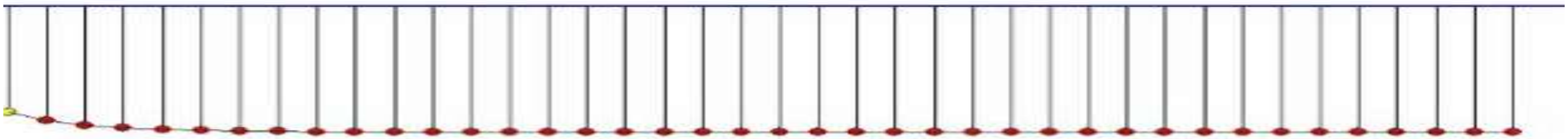
Solitary wave (Soliton) on pendula chain

Soliton (Solitary wave)

- Wave which maintains its shape and moves at constant speed
- Dispersion is balanced by nonlinear effects
- Nonlinear wave $f_{(a+b)} \neq f_{(a)} + f_{(b)}$
  (Principle of **superposition doesn't hold**)
- Behaves like "particle" (Localized)
- Different shapes, sizes and velocities



https://en.wikipedia.org/wiki/File:Soliton_hydro.jpg



PHYSICS OF CHAIN OF PENDULA

What is going on?

Are you satisfied with this explanation?

We aren't.

Example of Chain behavior (280 fps)

First pendulum is rotated

↓

Energy is stored to the deformation of springs

↓

1st pendulum undergoes 360°

↓

Kinetic energy is transferred along chain by springs

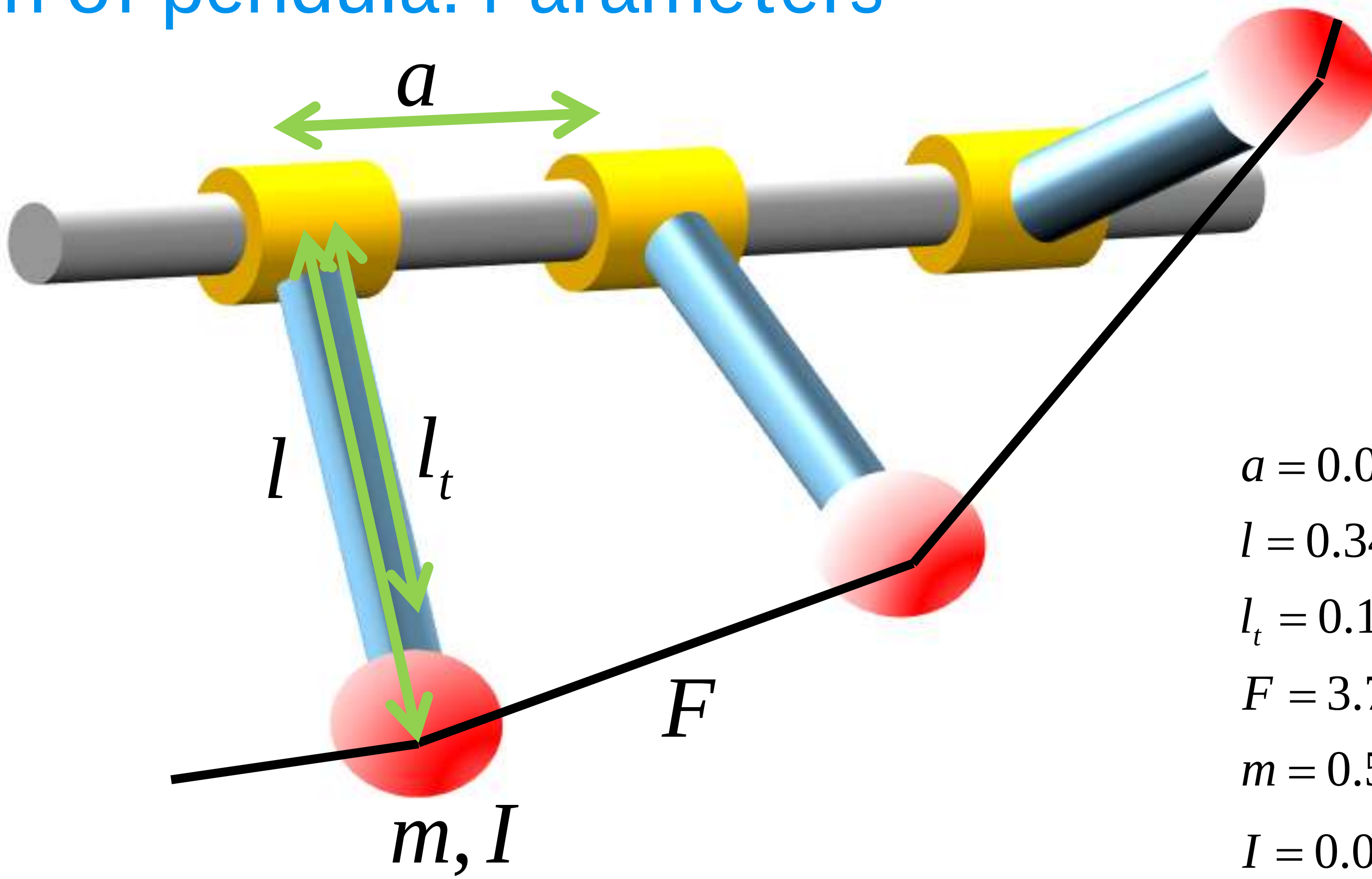
↓

Gravity returns pendula to the static state

↓

Reflection on Free end

Chain of pendula: Parameters



$a = 0.047m$

$l = 0.34m$

$l_t = 0.13m$

$F = 3.7N$

$m = 0.514kg$

$I = 0.004875kgm^2$

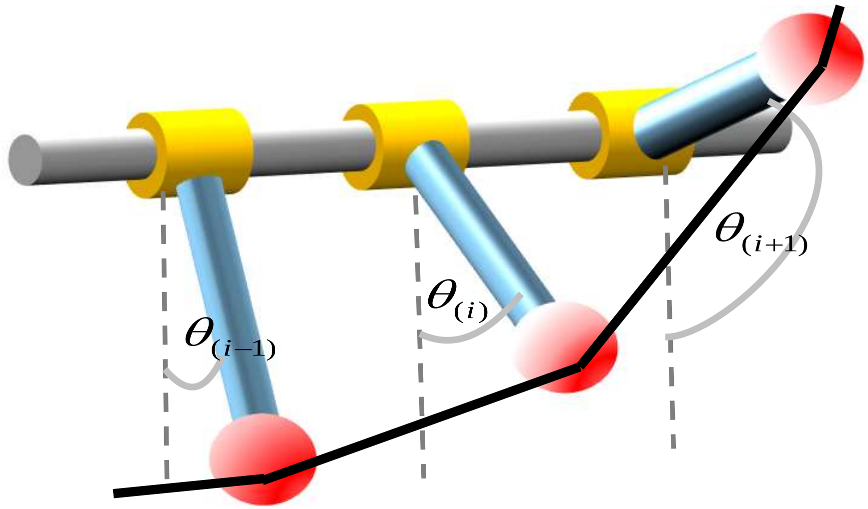
a	Spacing between pendula	F	String force of prestression
l	Length of pendulum	m	Mass of pendulum
l_t	Position of centre of mass	I	Moment of inertia around bar

Equation of motion

Kinetic energy $$T = \sum_{i=1}^{N} \frac{1}{2} I \dot{\theta}_i^2$$

Potential energy (Gravity) $$U_g = \sum_{i=1}^{N} mgl_t (1 - \cos(\theta_i))$$

Potential energy (Springs) $$U_s = \sum_{i=1}^{N-1} \frac{1}{2} \frac{Fl^2}{a} (\theta_i - \theta_{i+1})^2$$



Using Principle of least action

$$L = T - U_g - U_s$$

(Equivalent to the Force approach)

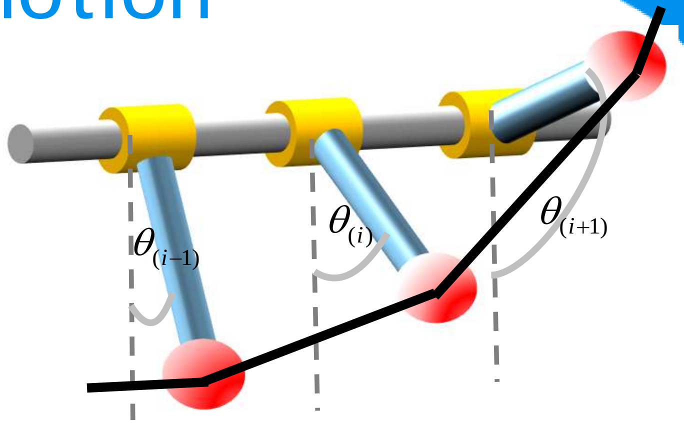
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} = 0$$

$$\ddot{\theta}_i = \frac{Fl^2}{Ia}(\theta_{i-1} + \theta_{i+1} - 2\theta_i) - \frac{mgl_t}{I}\sin(\theta_i)$$

Chain of pendula: Equation of motion

For discrete system

$$\ddot{\theta}_i = \frac{Fl^2}{Ia}\left(\theta_{i-1} + \theta_{i+1} - 2\theta_i\right) - \frac{mgl_t}{I}\sin(\theta_i)$$



From Torque analysis or Using principle of least action
(Derivation in appendices)

Continuous approximation to 1st order:

Using Taylor expansion:

$$\left(\theta_{i+1} + \theta_{i-1} - 2\theta_i\right) \approx a^2 \frac{\partial^2 \theta}{\partial x^2}$$

$$\frac{\partial^2 \theta}{\partial t^2} - c_0^2 \frac{\partial^2 \theta}{\partial x^2} + \omega_0^2 \sin\theta = 0$$

Sine-Gordon equation
Known analytical solution

a	Length between pendula
$c_0^2 = \frac{Fl^2 a}{I}$	Maximal possible information speed
$\omega_0^2 = \frac{mgl_t}{I}$	Natural frequency of single pendulum

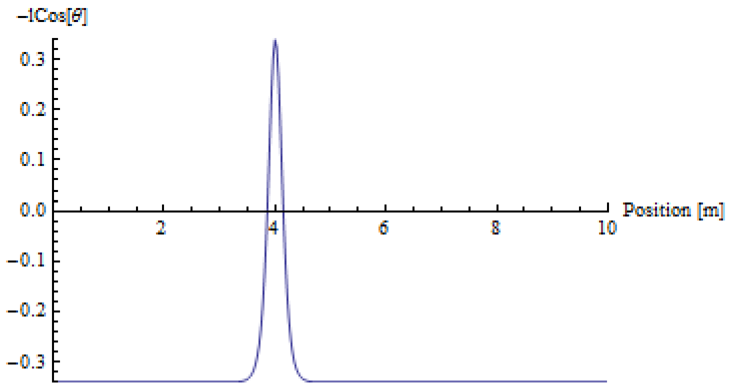
Analytic approach

Continuous system – Sine Gordon Equation

$$\underbrace{\frac{\partial^2\theta}{\partial t^2} - c_0^2\frac{\partial^2\theta}{\partial x^2}}_{\text{Terms of standard wave equation}} + \underbrace{\omega_0^2 \sin\theta}_{\text{Nonlinear term}} = 0$$

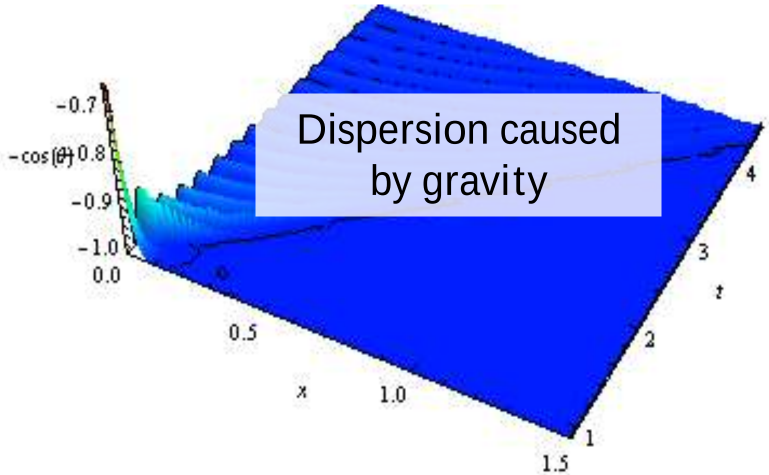
Terms of standard wave equation

Nonlinear term
Makes interesting phenomena

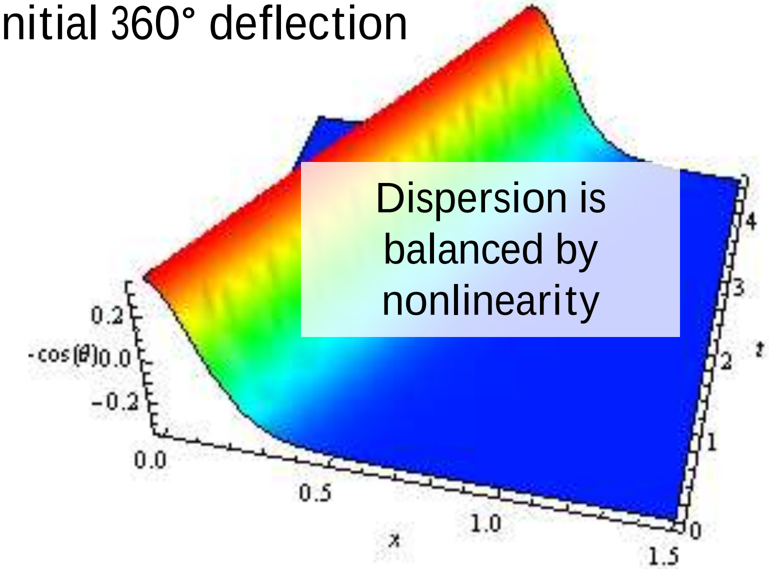


Predictions of continuous model (Sine-Gordon equation):

Initial deflections << 360°

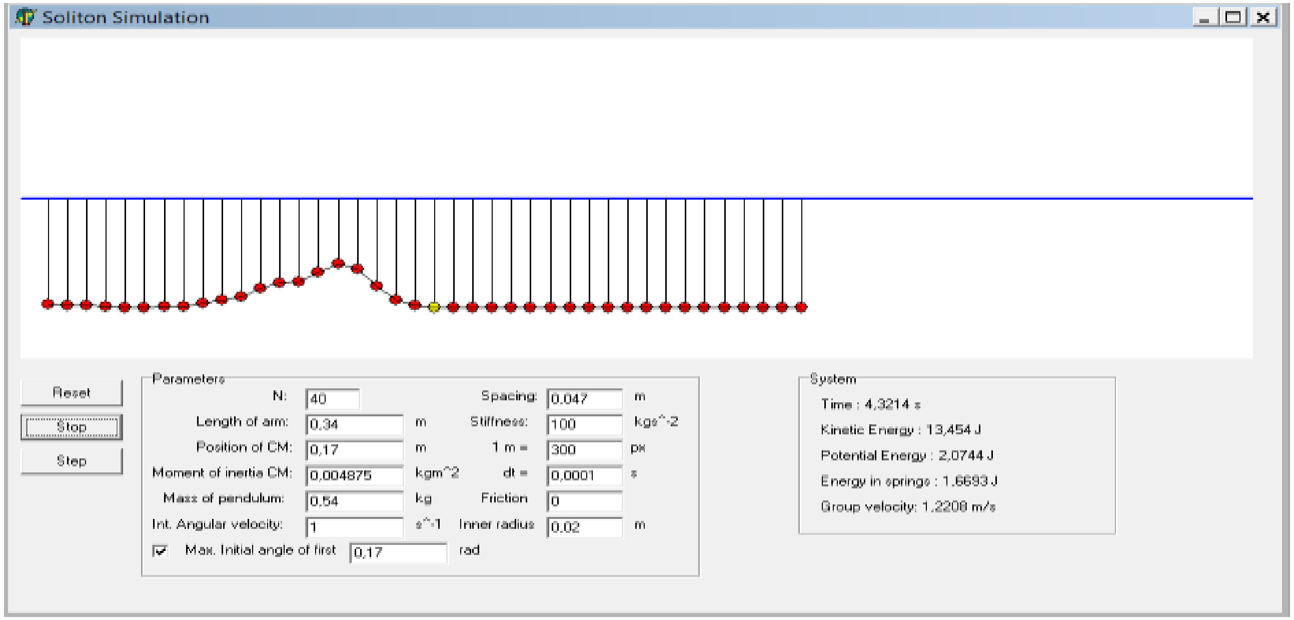


Initial 360° deflection



Discrete model: Numerical approach

Using Runge-Kutta
4th Order method



Without friction:

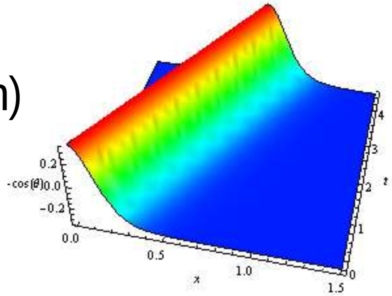
$$\ddot{\theta}_i = \frac{Fl^2}{Ia}(\theta_{i-1} + \theta_i - 2\theta_{i+1}) - \frac{mgl_t}{I}\sin(\theta_i)$$

With kinetic friction:

$$\ddot{\theta}_i = \frac{Fl^2}{Ia}(\theta_{i-1} + \theta_{i+1} - 2\theta_i) - \frac{mgl_t}{I}\sin(\theta_i) - \mathrm{sgn}(\dot{\theta})\frac{r}{I} f(mg + \frac{Fl}{Ia}(\theta_{i-1} + \theta_{i+1} - 2\theta_i))$$

Relations

Continuous model
(Sine-Gordon equation)



Discrete model
(Numerical simulation)

?

Experiment

PROPAGATION OF DEFLECTION

Small deflections

(<<360°)

Example of behaviour (<360°)

Dispersion result from analytical approach

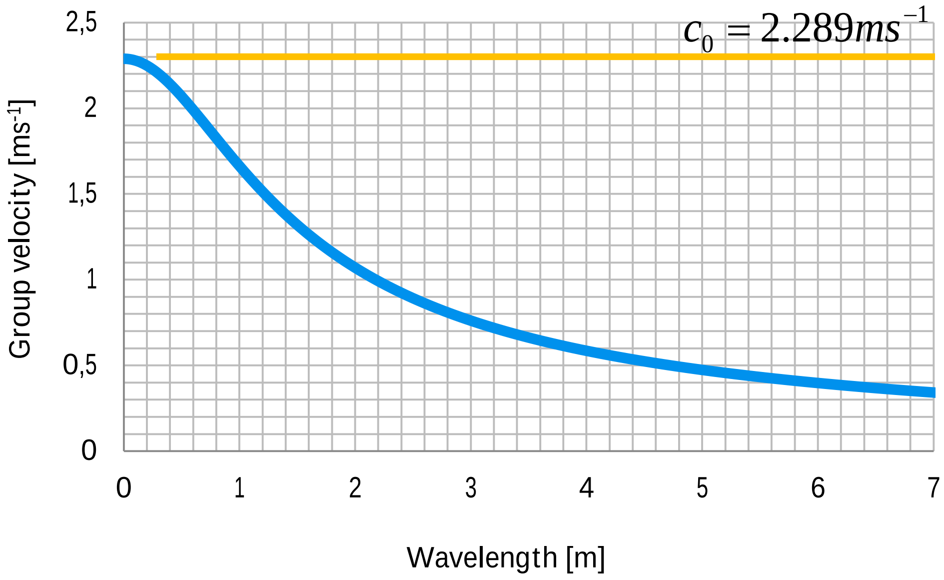
- Property of a system – Group velocity is wavelength depended

Dispersion law for our system (*Derivation in appendices)*

$$v_{group}(\lambda) = \frac{2\pi c_0^2}{\sqrt{\lambda^2\omega_0^2 + 4\pi^2 c_0^2}}$$

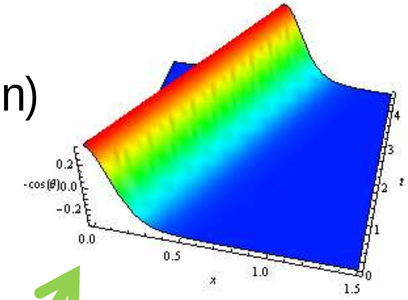
Theoretical prediction for our chain of pendula

From continuous model:



Relations

Continuous model
(Sine-Gordon equation)
Without friction

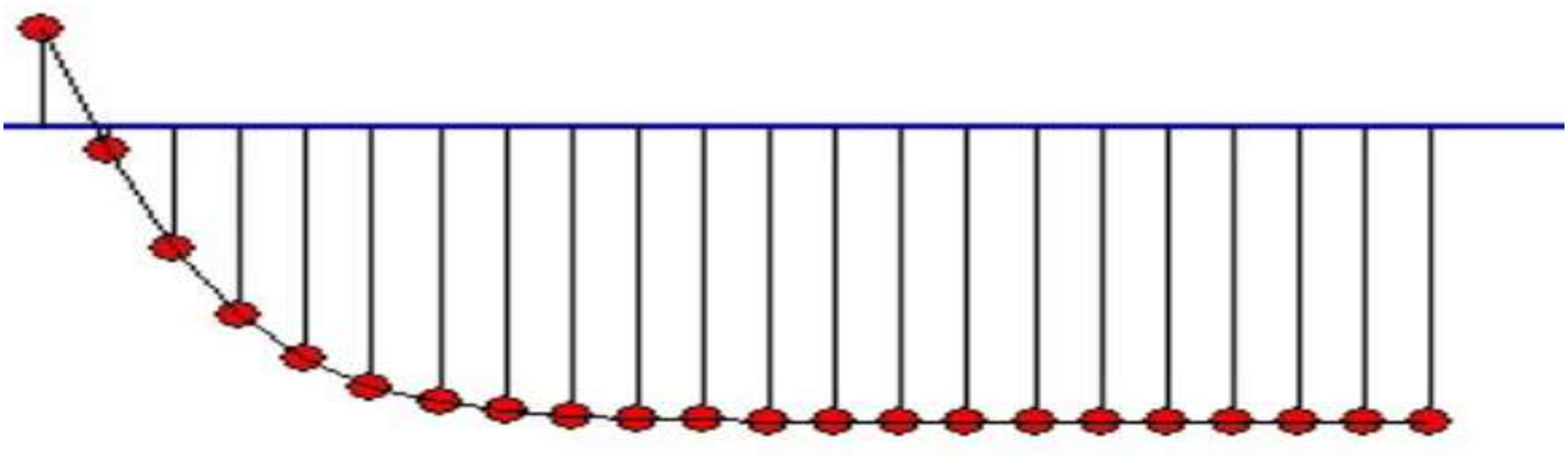


Discrete model
(Numerical simulation)

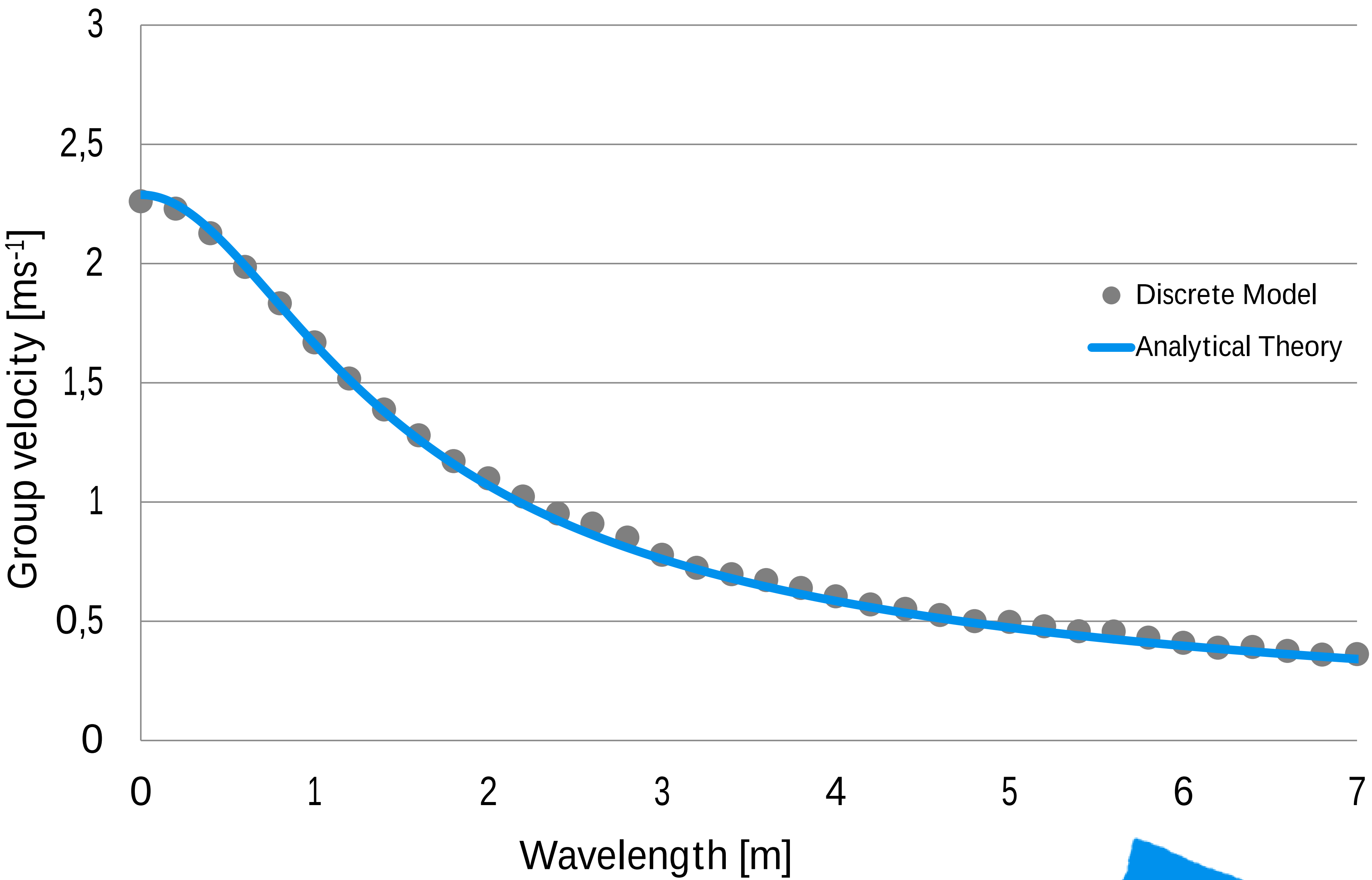
?

Experiment
Friction

Simulation of discrete system without friction

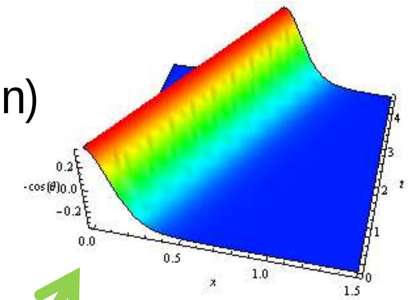


Analytical approach/Discrete model



Relations

Continuous model
(Sine-Gordon equation)
Without friction



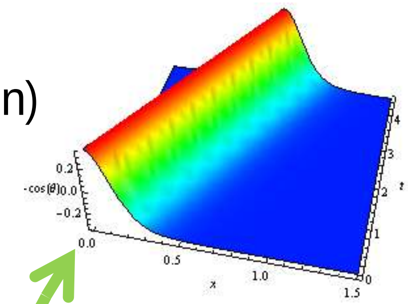
Discrete model
(Numerical simulation)

Without friction

Experiment
Friction

Relations

Continuous model
(Sine-Gordon equation)
Without friction



Discrete model
(Numerical simulation)

Without friction

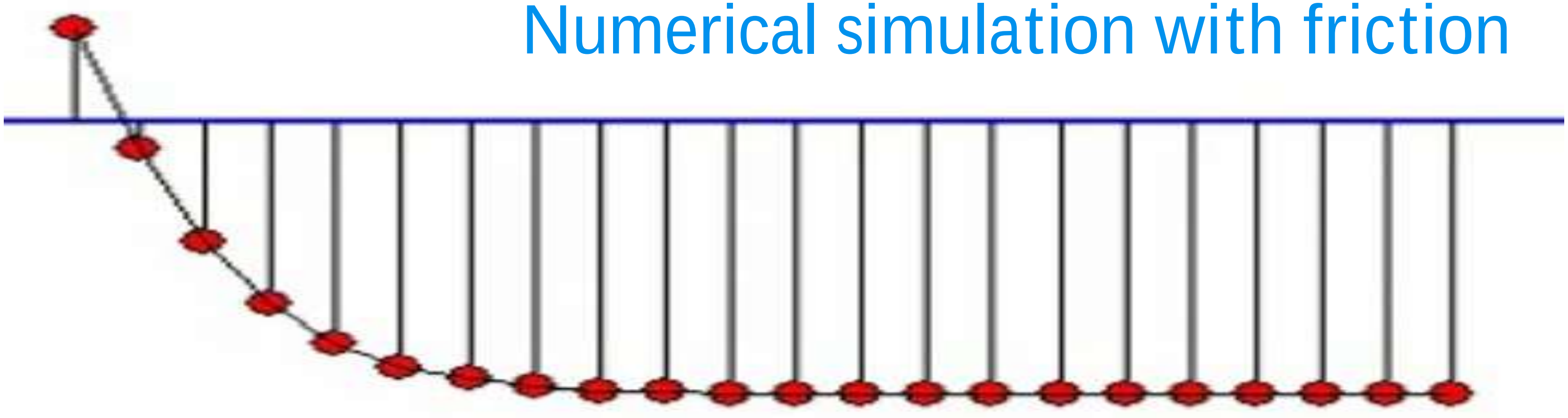
?

Experiment
Friction

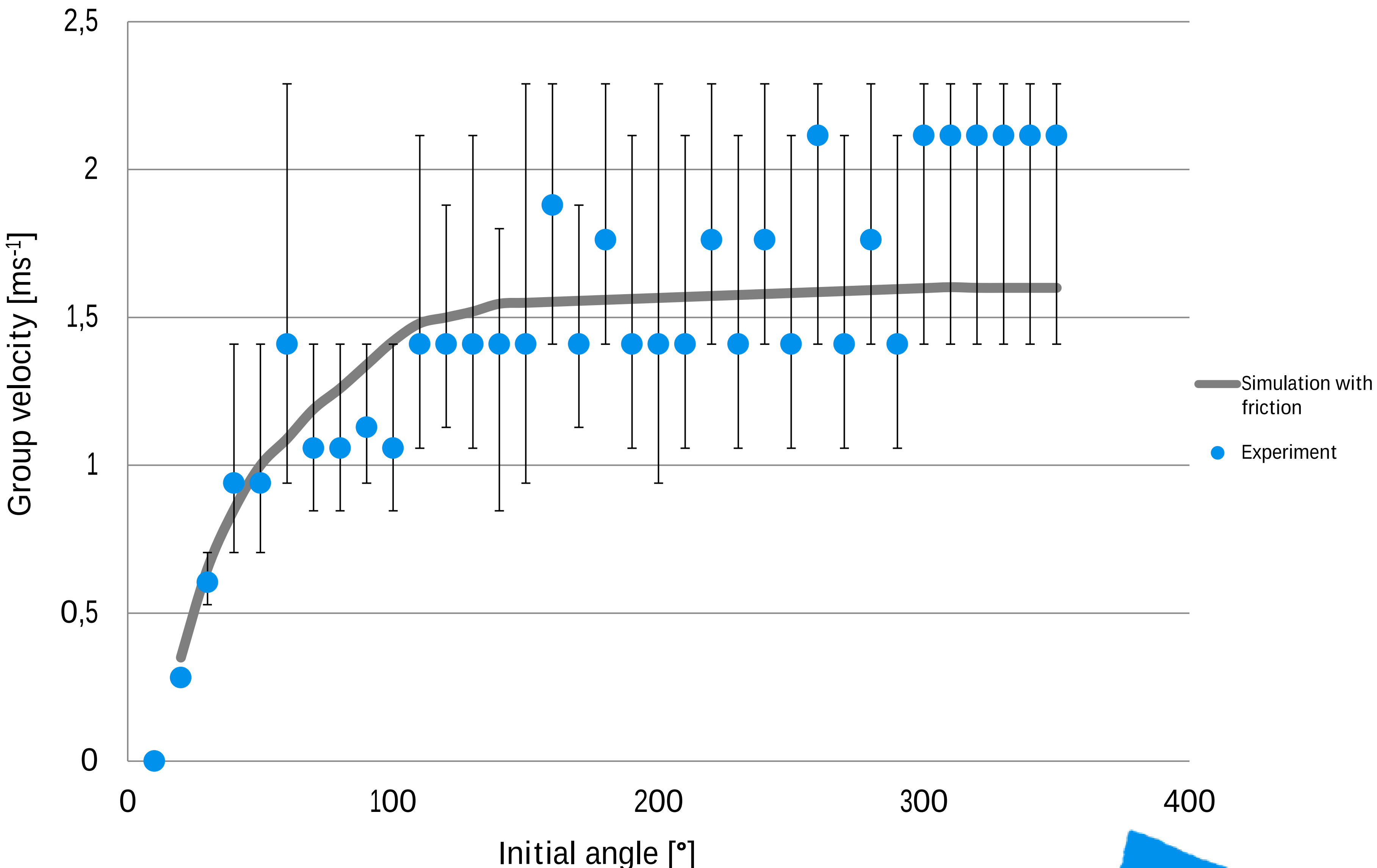
Comparison

Experiment

Numerical simulation with friction

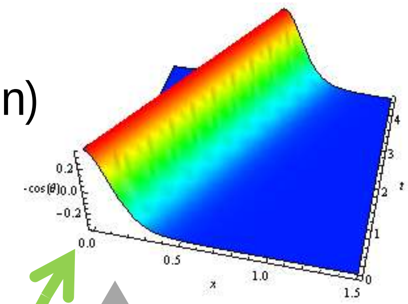


Group velocity / Initial angle of 1st pendulum



Relations

Continuous model
(Sine-Gordon equation)
Without friction



Discrete model
(Numerical simulation)

Without friction

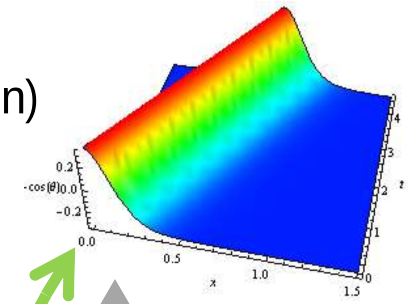
Friction

?

Experiment
Friction

Relations

Continuous model
(Sine-Gordon equation)
Without friction



Discrete model
(Numerical simulation)

Without friction

Friction

Friction cannot be included

Experiment
Friction

SOLITON

Entire 360° revolution

Propagation of deflection

Group velocity

What does Sine-Gordon eq. say ?

Continuous model:

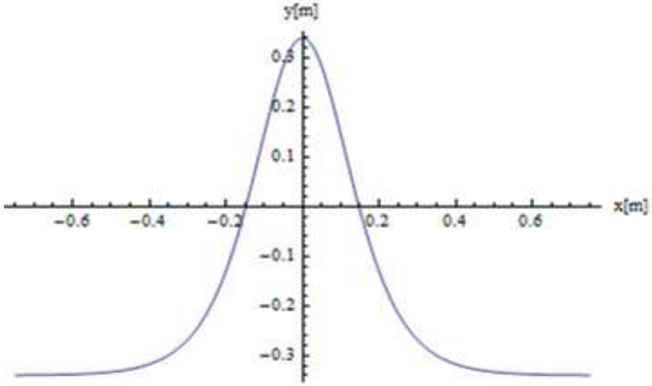
$$\frac{\partial^2\theta}{\partial t^2} - c_0^2 \frac{\partial^2\theta}{\partial x^2} + \omega_0^2 \sin\theta = 0$$

Solution with our boundary conditions[1]:

$$\theta_{(x,t)} = 4ArcTan\left(Exp\left(\frac{\omega_0 (x - vt)}{c_0\sqrt{1 - \frac{v^2}{c_0^2}}} \right) \right)$$

$$\omega_0 = \sqrt{\frac{mgl_t}{I}}$$

$$c_0 = \sqrt{\frac{Fa}{I}l}$$



What we actually see

$$y_{(x,t)} = -lCos(\theta_{(x,t)})$$

Still one unknown variable v - Group velocity !

Related to initial angular velocity at maximum *(Derivation in appendices)*

$$|v| = \sqrt{\frac{\omega_{initial}^2}{4\omega_0^2 + \omega_{initial}^2}} c_0$$

[1] Physics of solitons. M. Peyrard, T. Dauxios, *Cambridge University Press (2010)*

Interesting properties

$$|v| = \sqrt{\frac{{\omega_{initial}}^2}{4{\omega_0}^2 + {\omega_{initial}}^2}}\, c_0$$

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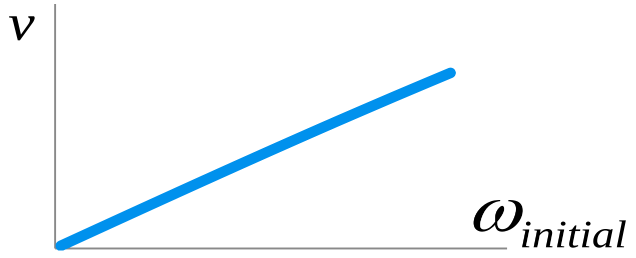
Small initial angular velocities

$$\omega_0 >> \omega_{initial} \qquad v \approx \frac{\omega_{initial}}{2\omega_0} c_0$$



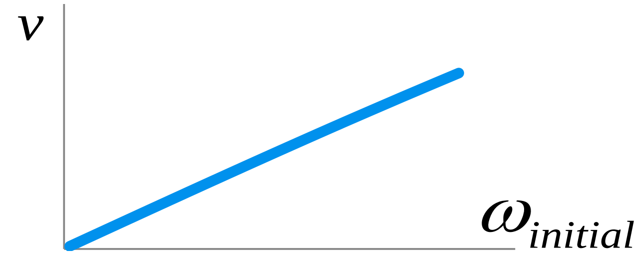
Group velocity rises linearly

Interesting properties

$$|v| = \sqrt{\frac{\omega_{initial}^2}{4\omega_0^2 + \omega_{initial}^2}} c_0$$

Small initial angular velocities

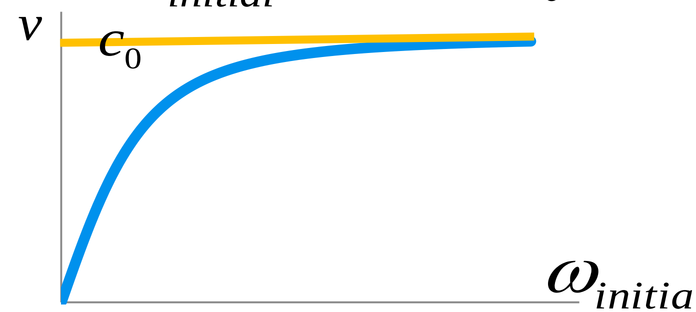
$\omega_0 >> \omega_{initial}$ $\quad v \approx \frac{\omega_{initial}}{2\omega_0} c_0$



Group velocity rises linearly

Large initial angular velocities

$\omega_0 << \omega_{initial}$ $\quad v \to c_0$



Group velocity converges to maximal value

Interesting properties

$$|v| = \sqrt{\frac{\omega_{initial}^2}{4\omega_0^2 + \omega_{initial}^2}}\, c_0$$

Small initial angular velocities

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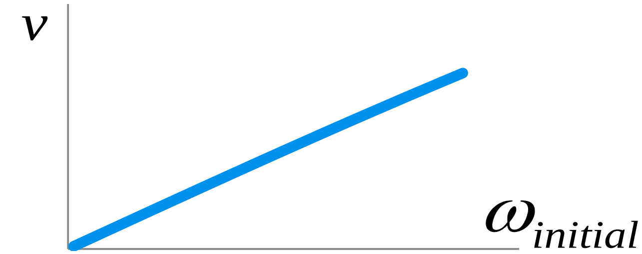
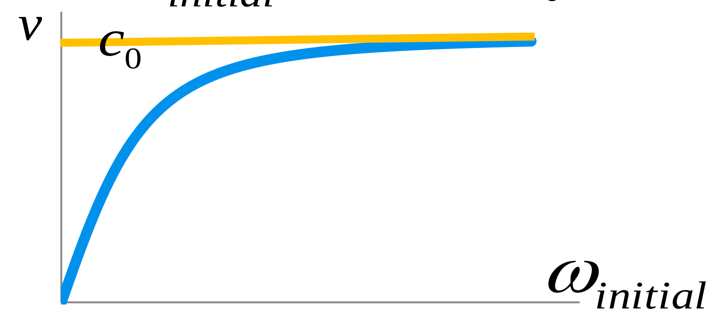


Group velocity rises linearly

Large initial angular velocities

$$\omega_0 << \omega_{initial} \qquad v \to c_0$$



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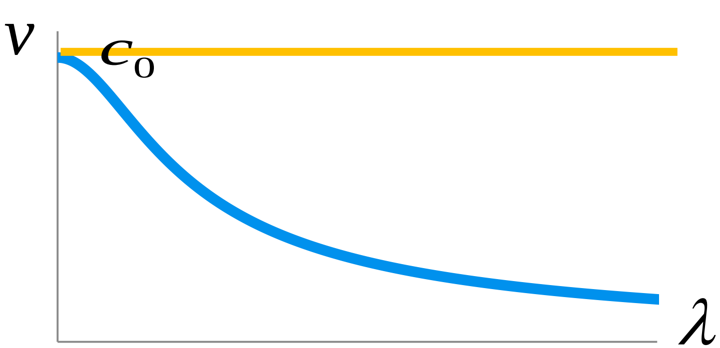


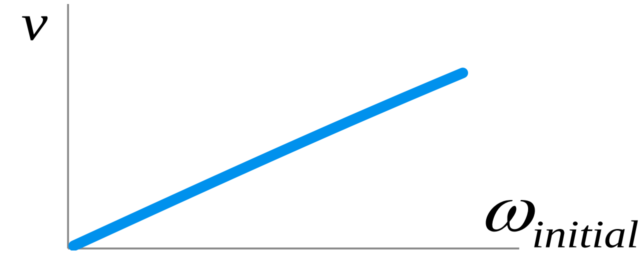
Dispersion law for small deflections

Interesting properties

$$|v| = \sqrt{\frac{\omega_{initial}^2}{4\omega_0^2 + \omega_{initial}^2}} c_0$$

Small initial angular velocities

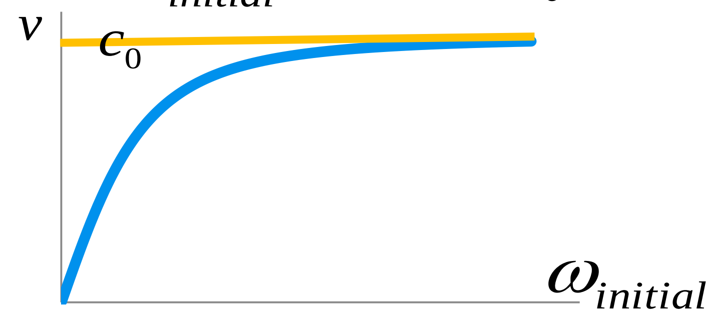
$$\omega_0 >> \omega_{initial} \qquad v \approx \frac{\omega_{initial}}{2\omega_0} c_0$$



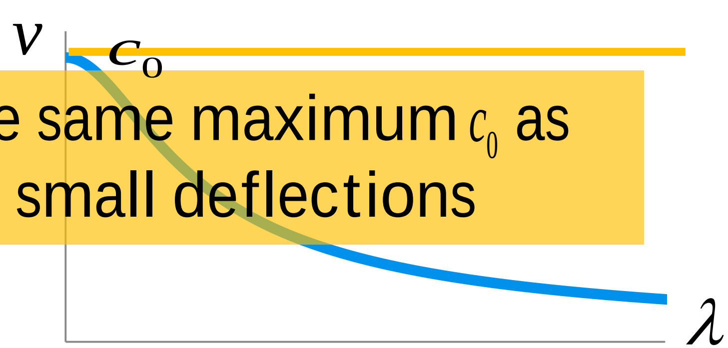
Group velocity rises linearly

Large initial angular velocities

$$\omega_0 << \omega_{initial} \qquad v \to c_0$$



Group velocity converges to maximal value



The same maximum c_0 as for small deflections

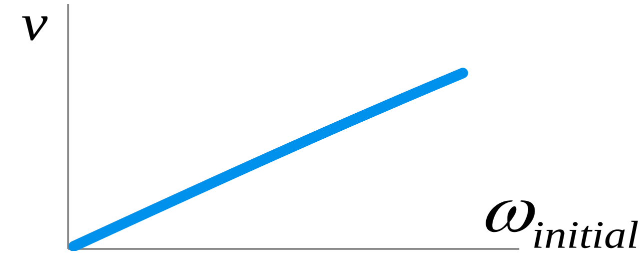
Dispersion law for small deflections

Interesting properties

$$|v| = \sqrt{\frac{\omega_{initial}^2}{4\omega_0^2 + \omega_{initial}^2}} c_0$$

Small initial angular velocities

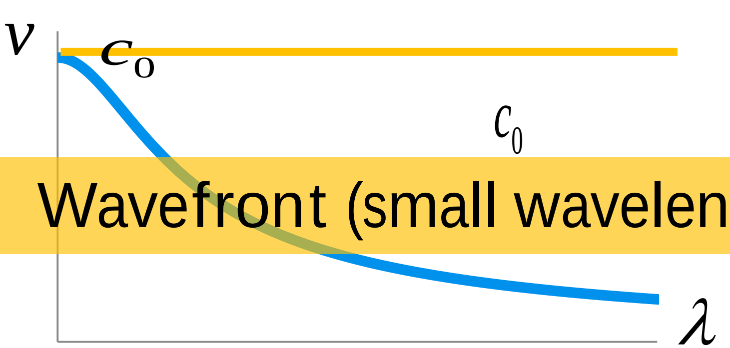
$$\omega_0 >> \omega_{initial} \qquad v \approx \frac{\omega_{initial}}{2\omega_0} c_0$$



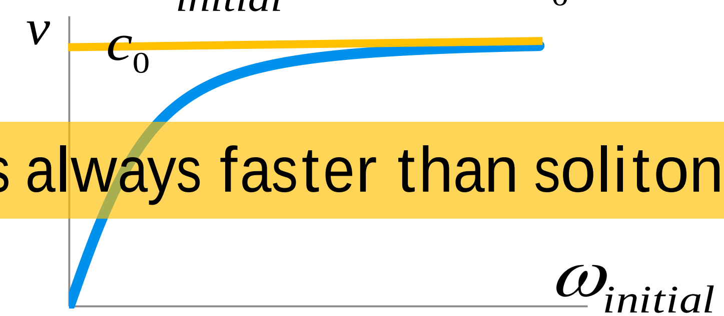
Group velocity rises linearly

Large initial angular velocities

$$\omega_0 << \omega_{initial} \qquad v \to c_0$$



Dispersion law for small deflections

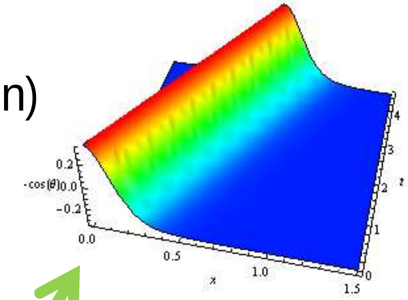


Group velocity converges to maximal value

Wavefront (small wavelengths) is always faster than soliton

Relations

Continuous model
(Sine-Gordon equation)
Without friction



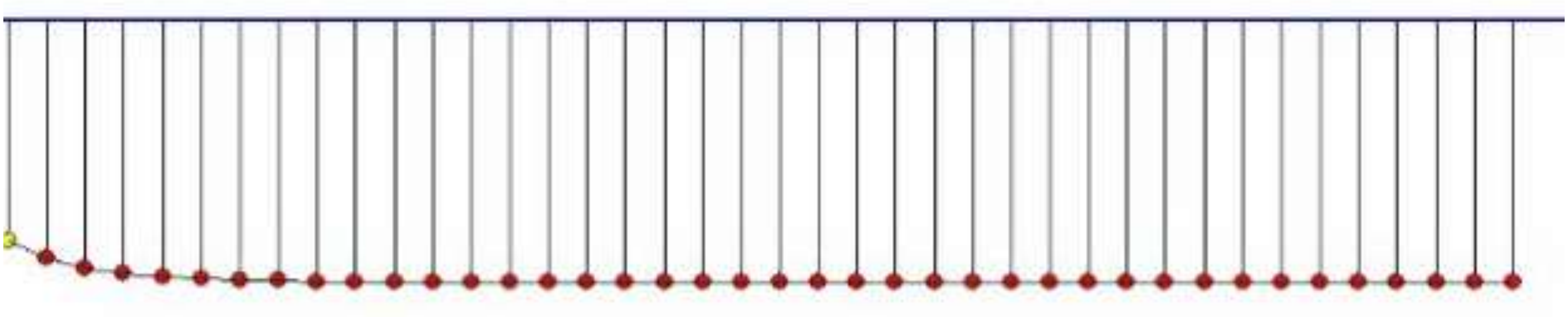
Discrete model
(Numerical simulation)

?

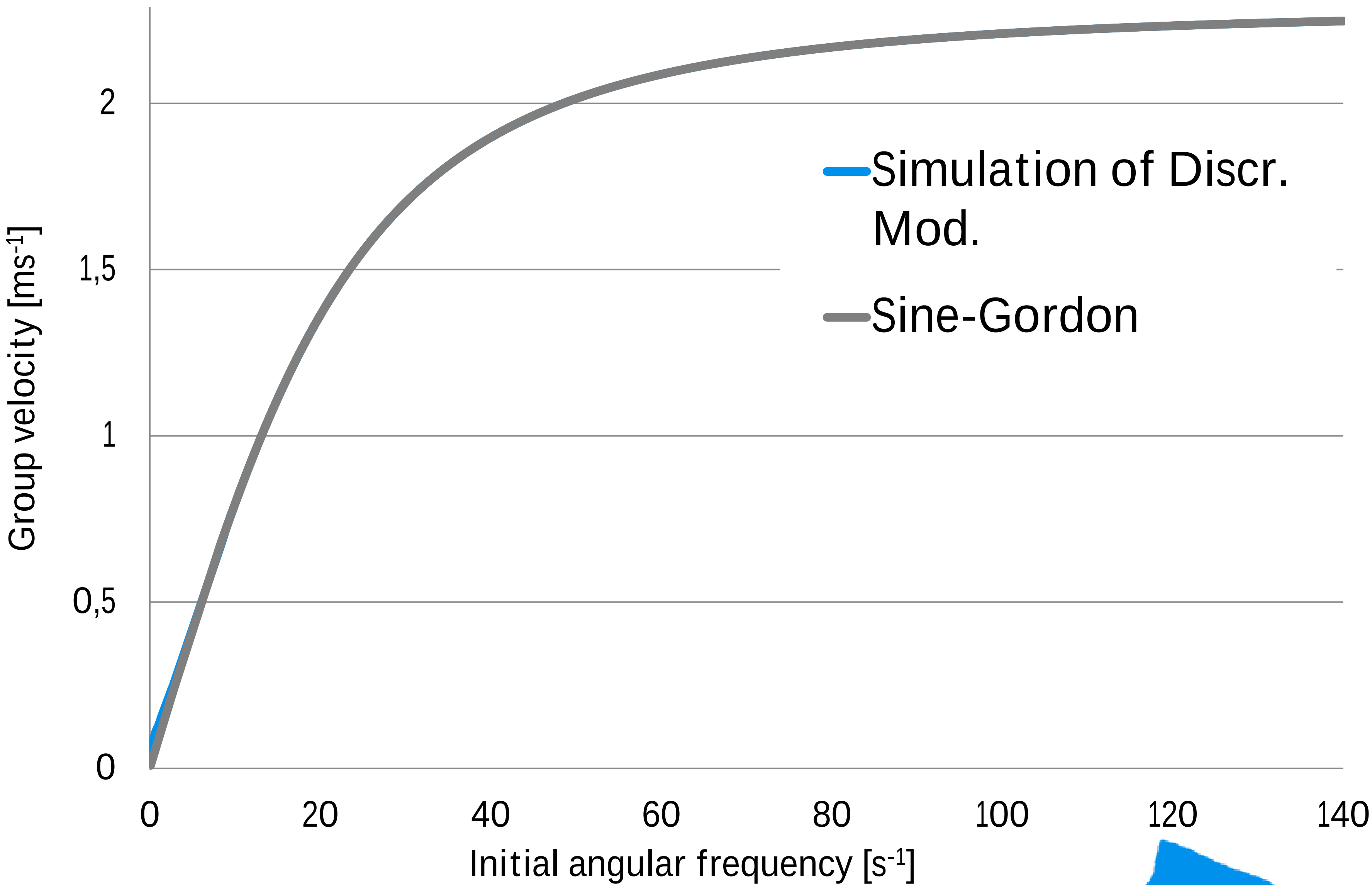
Experiment
Friction

Simulation of discrete system without friction

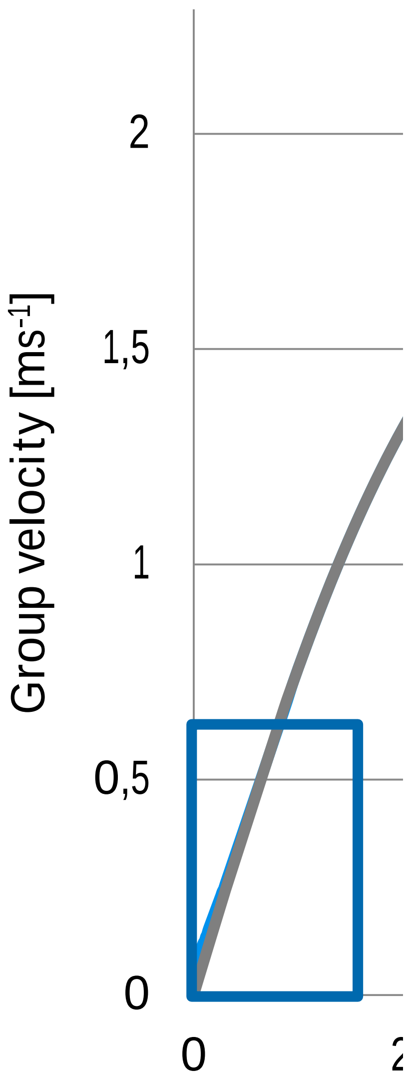
Maintains its shape and moves at constant speed → Soliton

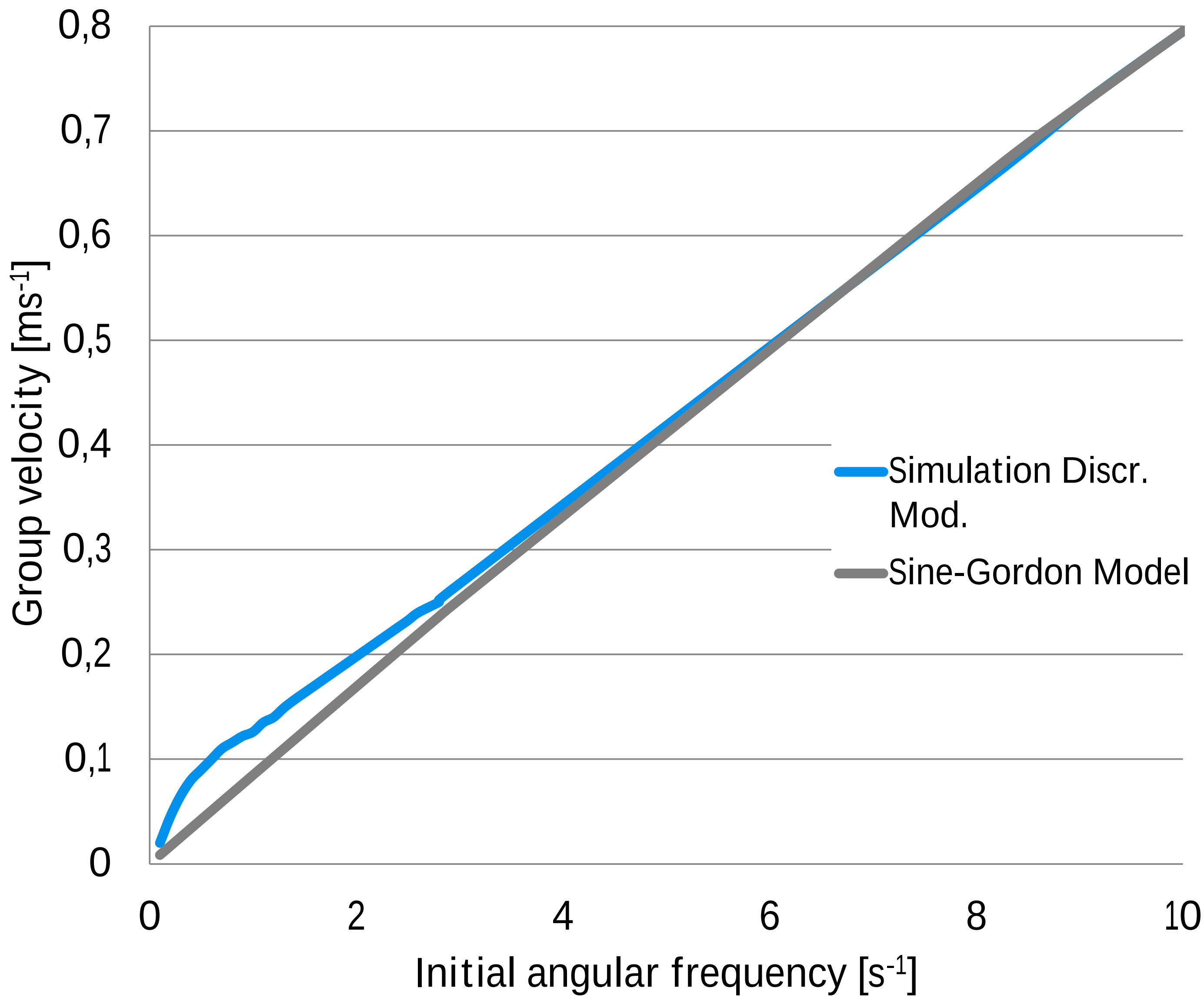


Group velocity / Initial angular freq.



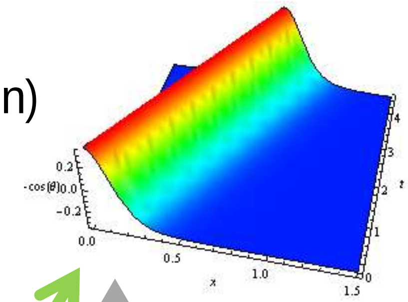
Group velocity / Initial angular freq.





Relations

Continuous model
(Sine-Gordon equation)
Without friction



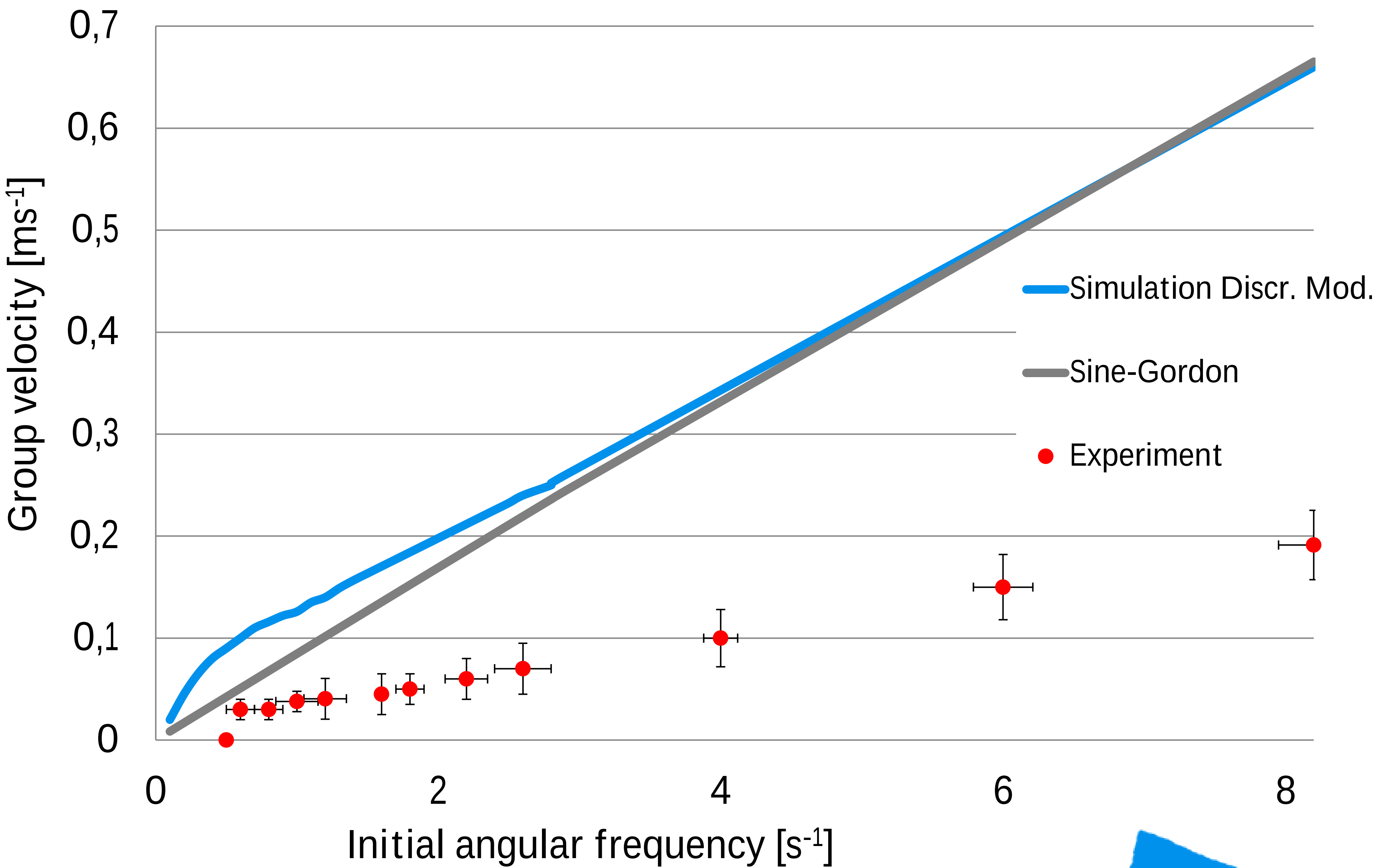
Discrete model
(Numerical simulation)

Without friction

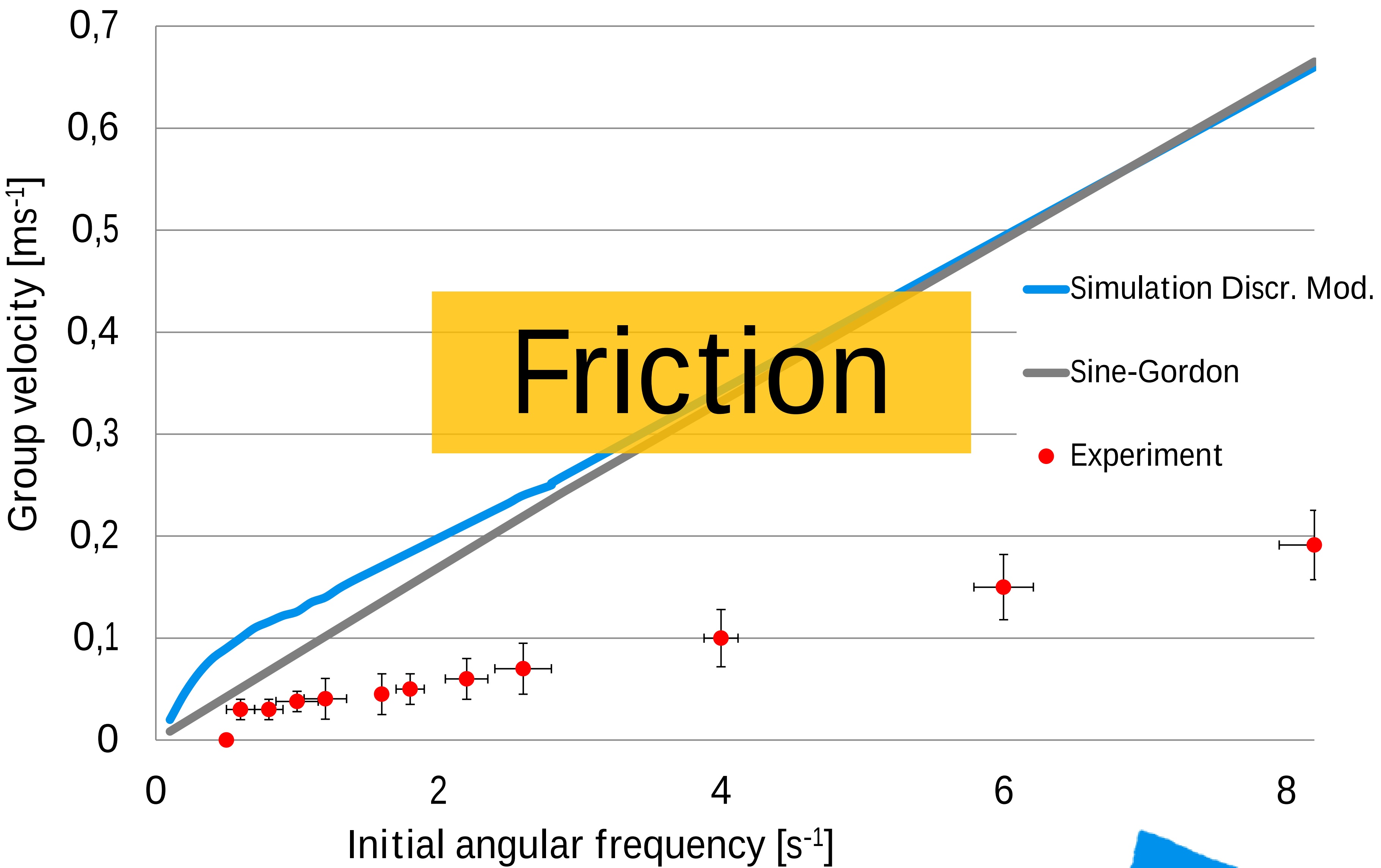
?

Experiment
Friction

Group velocity / Initial angular freq.

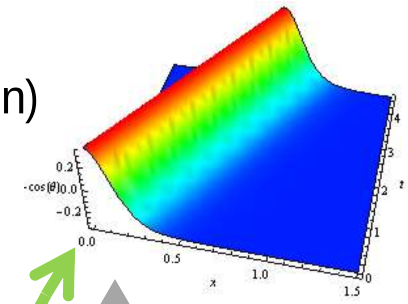


Group velocity / Initial angular freq.



Relations

Continuous model
(Sine-Gordon equation)
Without friction



Discrete model
(Numerical simulation)

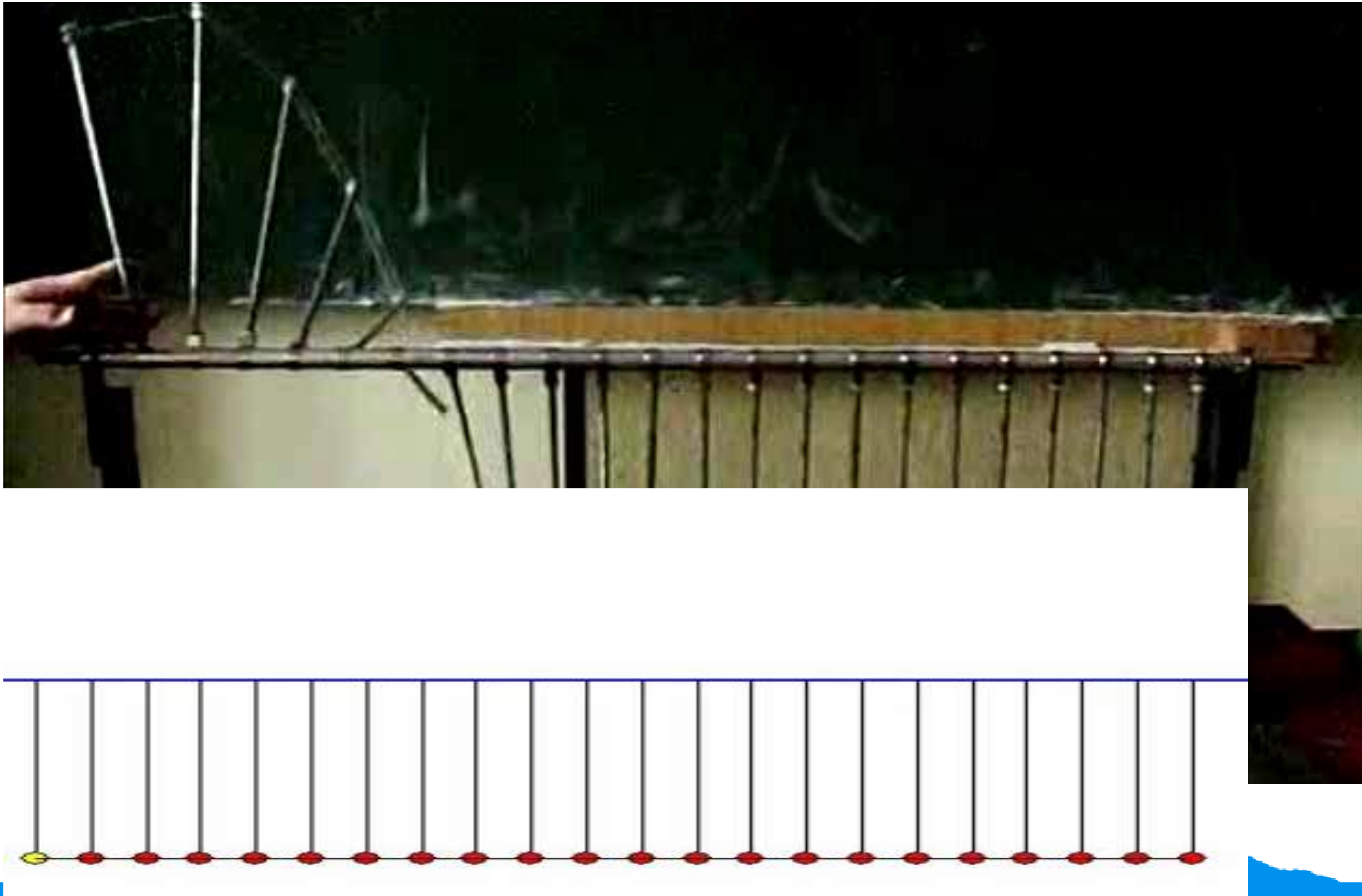
Without friction

Friction cannot be included

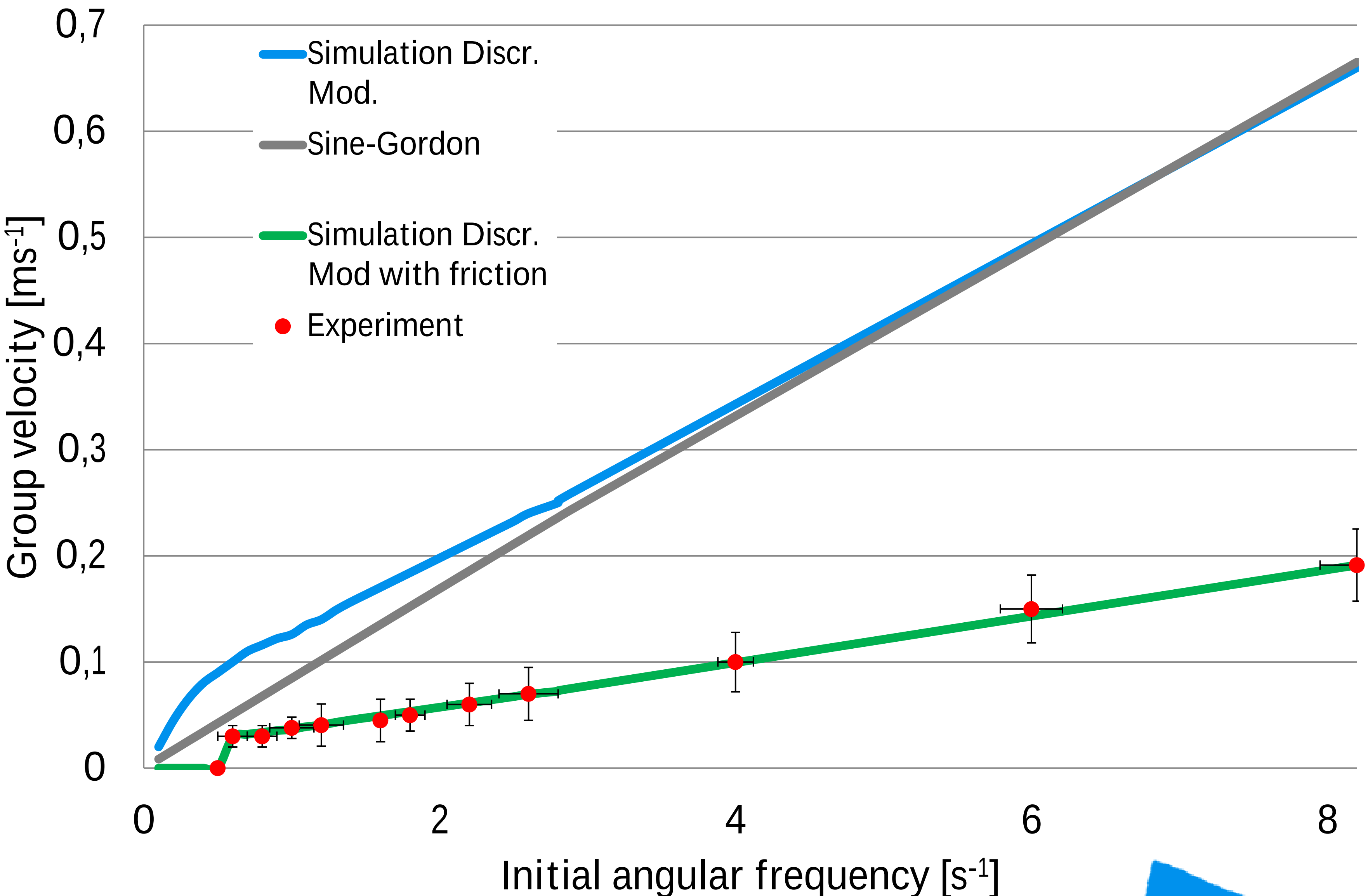
?

Experiment
Friction

Experiment
Comparison

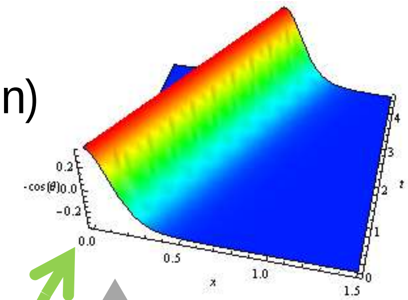


Group velocity / Initial angular frequency



Relations

Continuous model
(Sine-Gordon equation)
Without friction



Discrete model
(Numerical simulation)

Without friction

Friction

Friction cannot be included

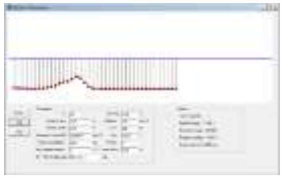
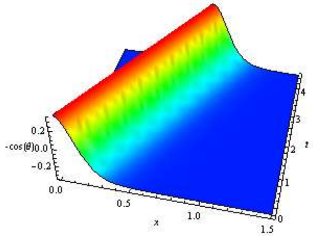
Experiment
Friction

SUMMARY

Conclusion

- Analysis of the system & Equation of motion

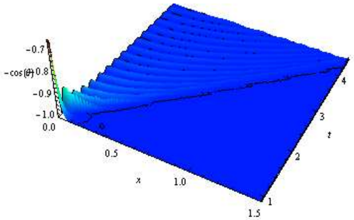
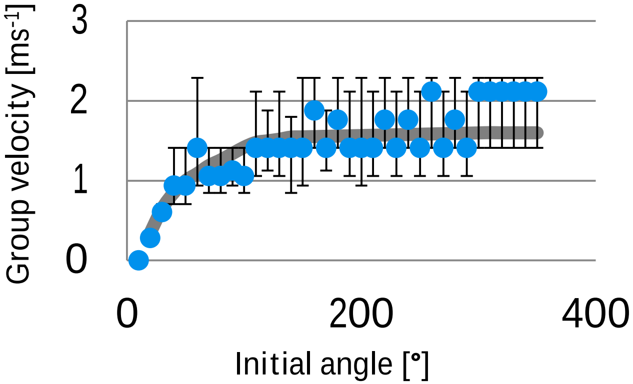
$$\ddot{\theta}_i = \frac{Fl^2}{Ia}(\theta_{i-1} + \theta_{i+1} - 2\theta_i)) - \frac{mgl_t}{I}\sin(\theta_i)$$

Experiment & Numerical simulation & Sine-Gordon

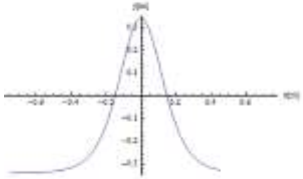
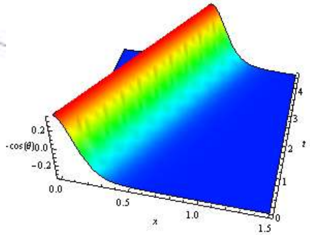
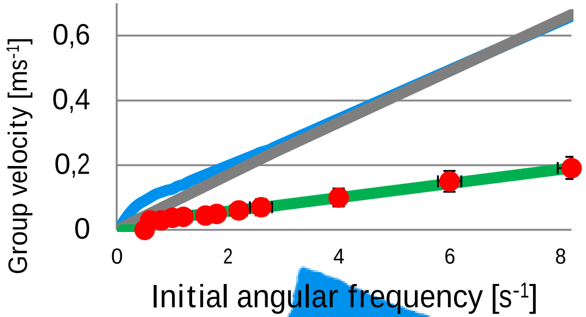
- Propagation of deflection
  - Small deflections (<< 360°)

Dispersion & Dissipation of profile

  - Soliton

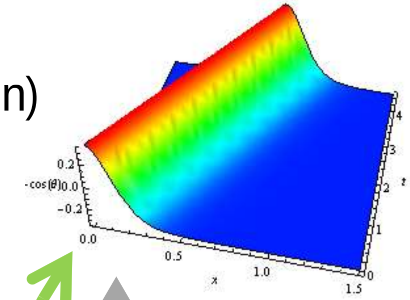
Stable against dispersion, but not against friction

Conclusion

- Small deflections (<< 360°)
- Soliton

Continuous model
(Sine-Gordon equation)
Without friction



Discrete model
(Numerical simulation)

Without friction

Friction

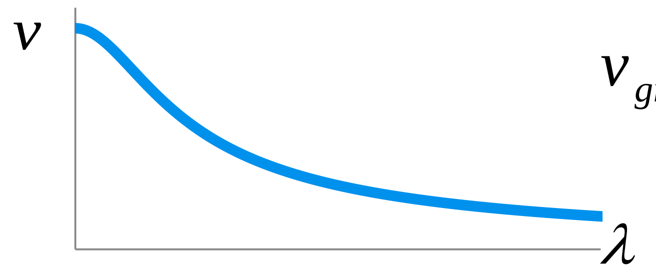
Friction cannot be included

Experiment
Friction

Conclusion

Thank you for your attention!

- Small deflections (<< 360°)

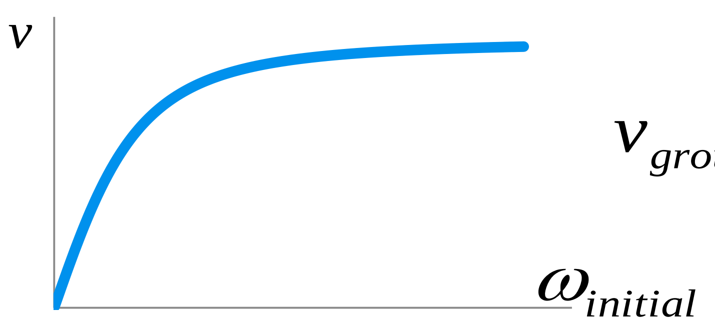


$$v_{group} = \frac{2\pi c_0^2}{\sqrt{\lambda^2 \omega_0^2 + 4\pi^2 c_0^2}}$$

Dispersion law for small deflections

Wavefront (small wavelengths) is always faster than soliton

- Soliton



$$v_{group} = \sqrt{\frac{\omega_{initial}^2}{4\omega_0^2 + \omega_{initial}^2}}\, c_0$$

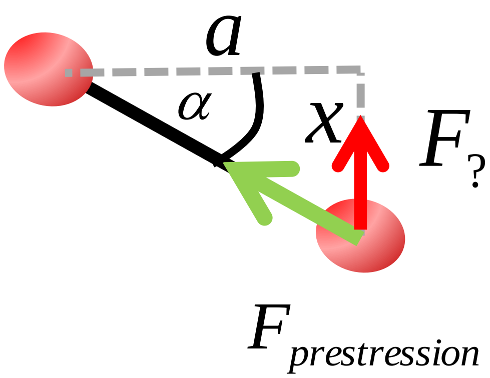
Group velocity converges to maximal value

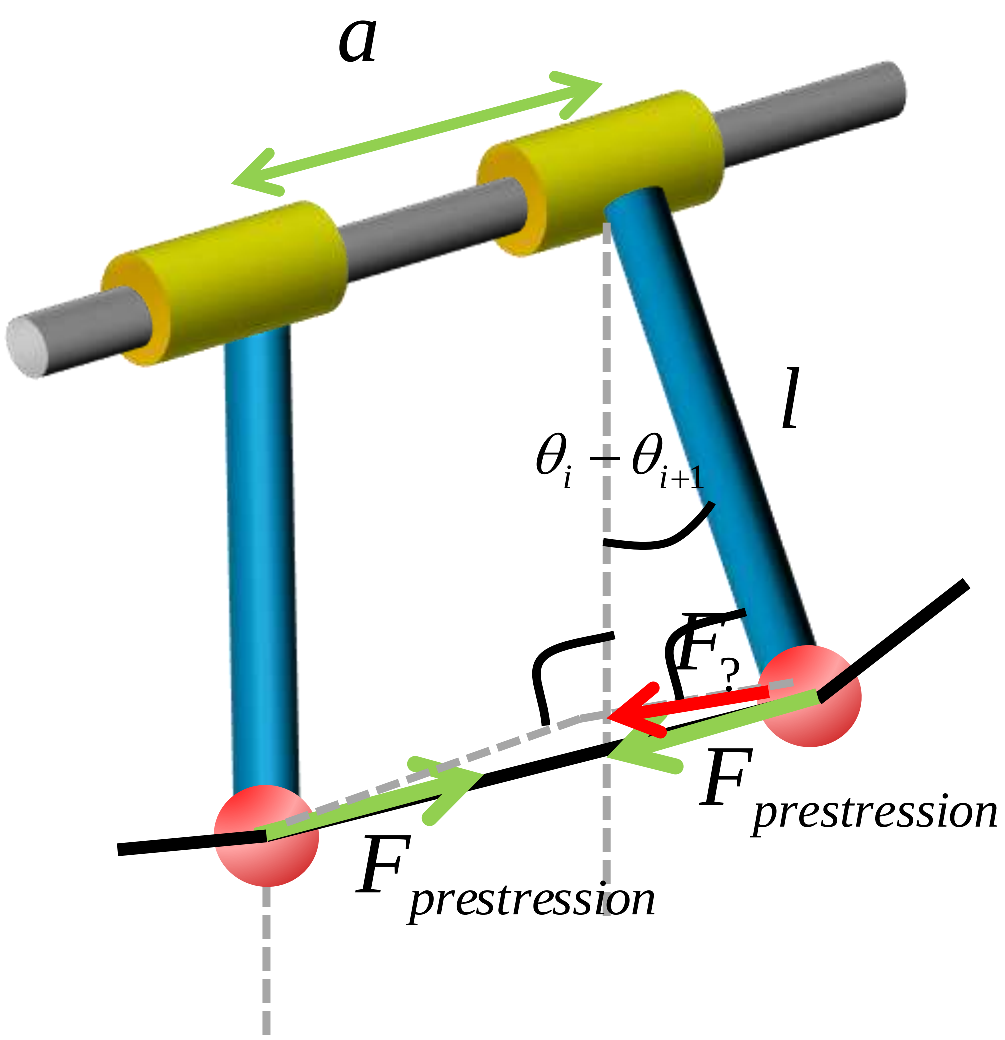
APPENDICES

Energy in springs

Looking from reference frame connected with previous pendulum

View from above:

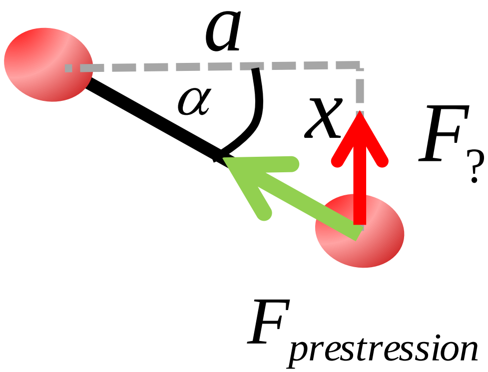




Energy in springs

Looking from reference frame connected with previous pendulum

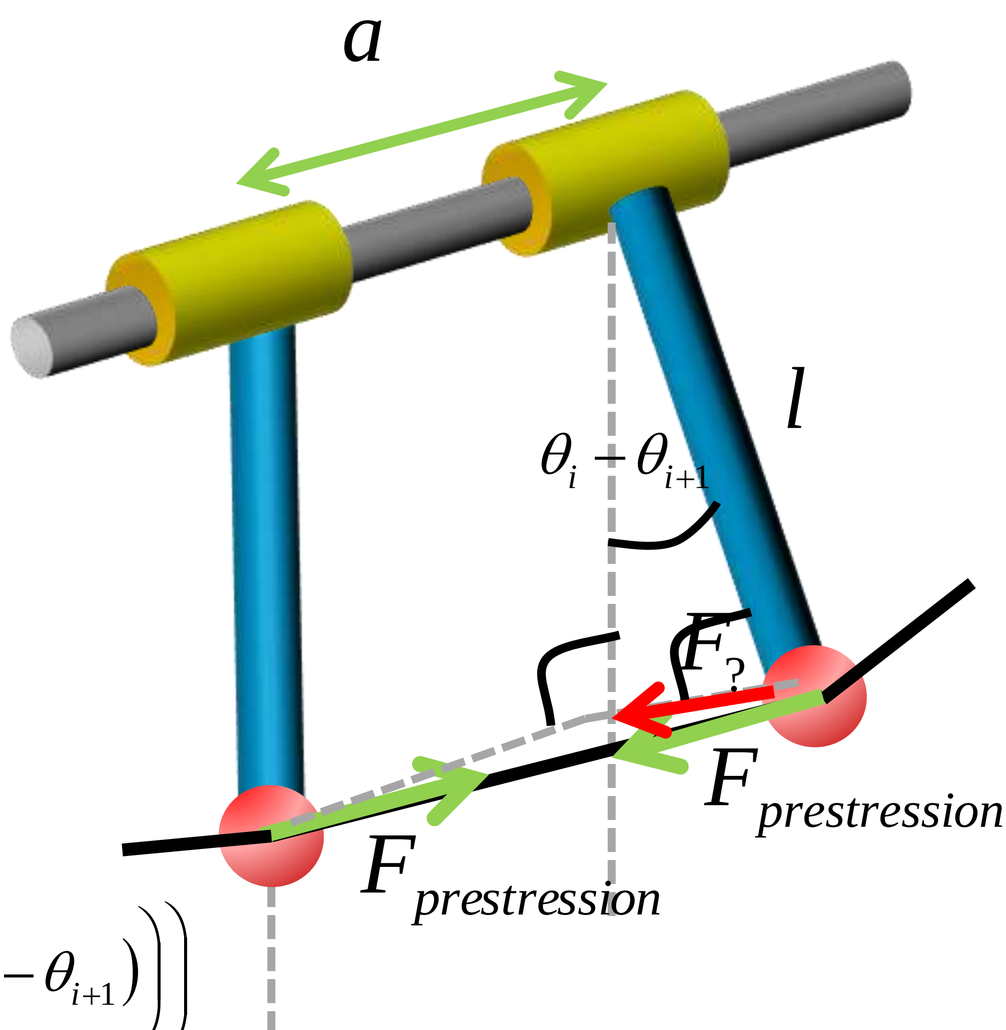
View from above:



$$x = l\tan(\theta_i - \theta_{i+1})$$

$$\tan\alpha = \frac{x}{a}$$

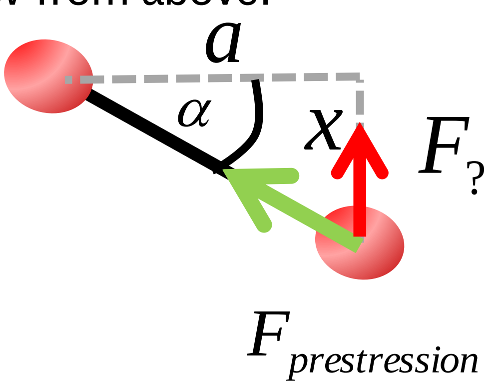
$$F_? = F_{prestression}\sin\left(\arctan\left(\frac{l}{a}\tan(\theta_i - \theta_{i+1})\right)\right)$$

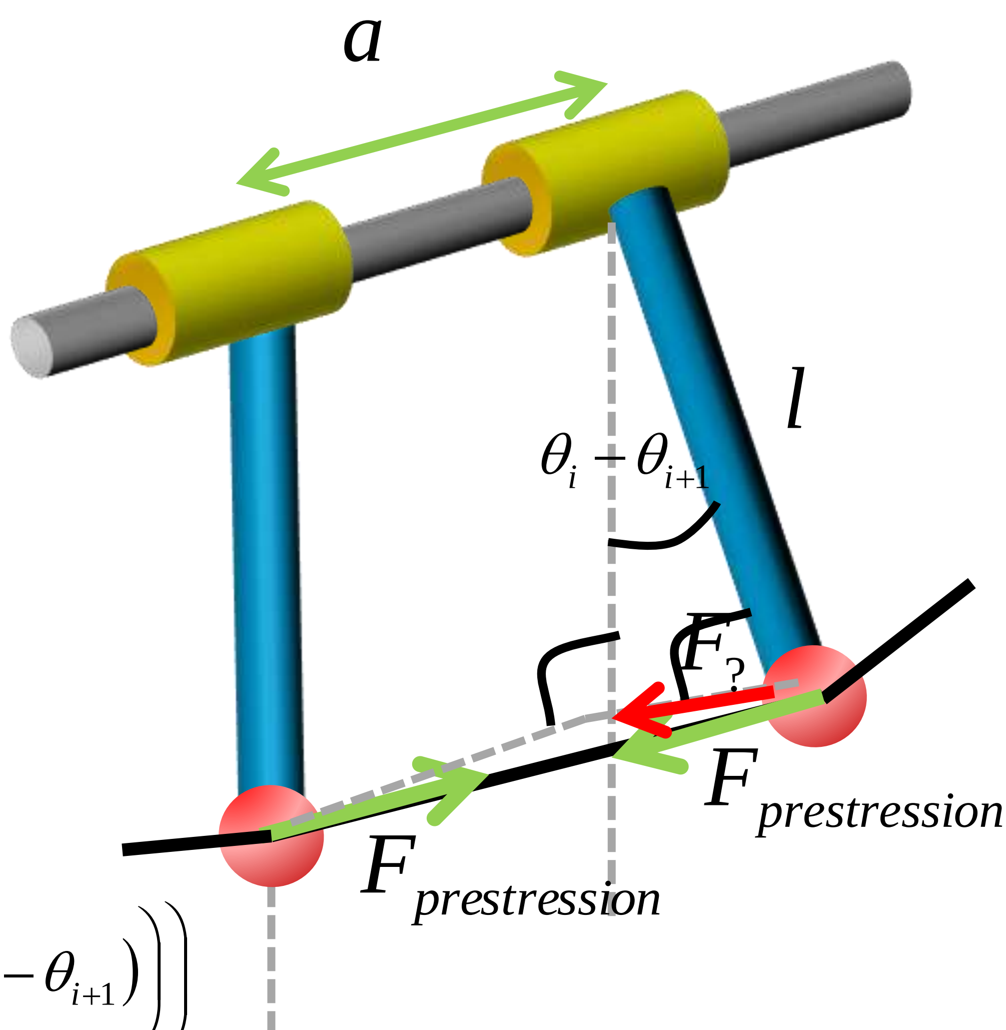


Energy in springs

Looking from reference frame connected with previous pendulum

View from above:





$$x = l\tan(\theta_i - \theta_{i+1})$$

$$\tan\alpha = \frac{x}{a}$$

$$F_? = F_{prestression}\sin\left(\arctan\left(\frac{l}{a}\tan(\theta_i - \theta_{i+1})\right)\right)$$

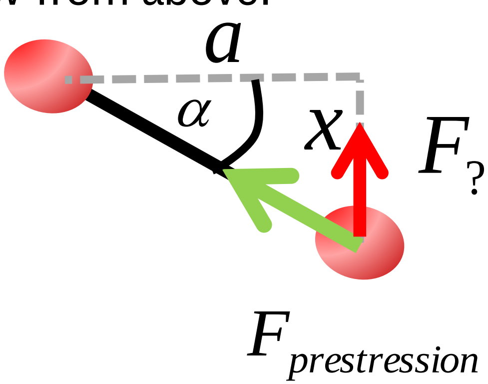
Assuming strong spring $\rightarrow$ small $\theta_i - \theta_{i+1}$

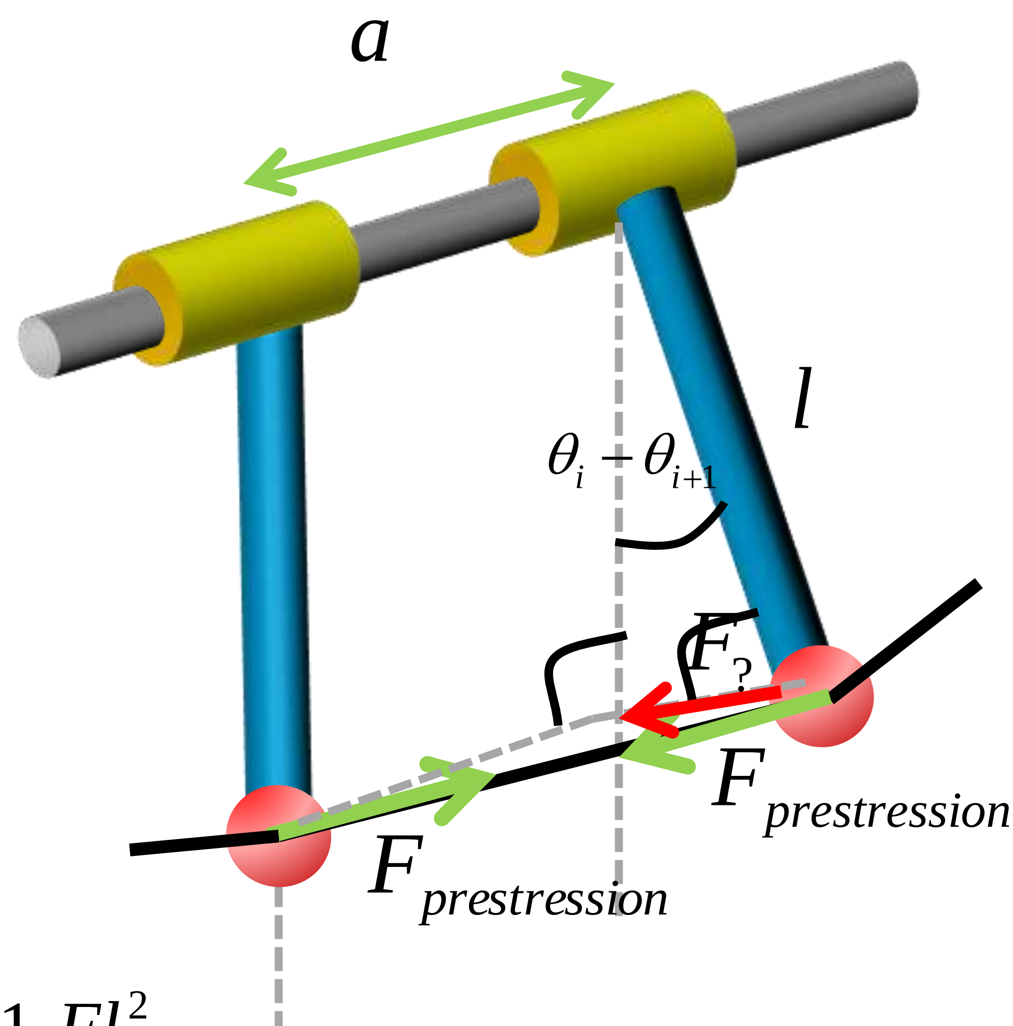
$$F_? = F_{prestression}\frac{l}{a}(\theta_i - \theta_{i+1}) + O(F^3)$$

Energy in springs

Looking from reference frame connected with previous pendulum

View from above:

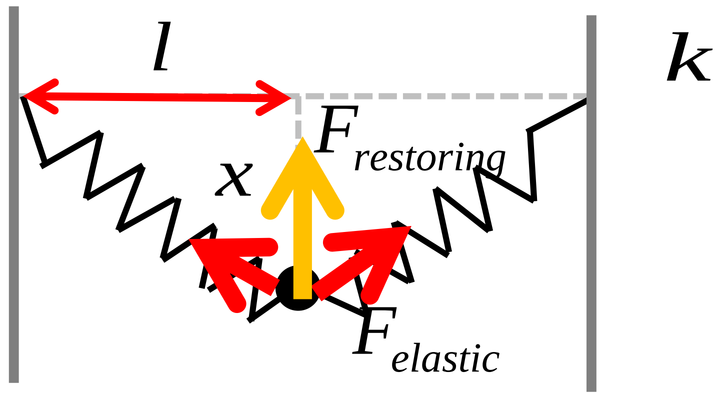




$$E=\int F_? dx=\int F_? l d(\Delta\theta)=\frac{1}{2}\frac{Fl^2}{a}(\theta_i-\theta_{i+1})^2+O((\theta_i-\theta_{i+1})^4)$$

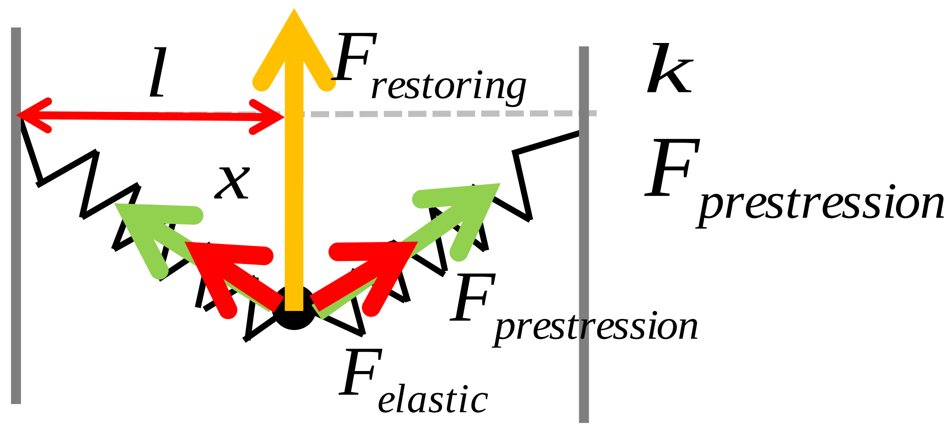
Why prestression?

Usual spring



$$F_{elastic} = k(\sqrt{x^2 + l^2} - l^2) \approx \frac{1}{2} k \frac{x^2}{l} + O(x^4)$$

Prestressed spring



$$F_{elastic} = k(\sqrt{x^2 + l^2} - l^2) \approx \frac{1}{2} k \frac{x^2}{l} + O(x^4)$$

$$F_{prestression}$$

Usual spring $$F_{restoring} = 2F_{elastic} \frac{x}{l} \approx k \frac{x^3}{l^2} + O(x^5)$$

Prestressed spring $$F_{restoring} = 2(F_{prestression} + F_{elastic}) \frac{x}{l} \approx 2F_{prestression} \frac{x}{l} + k \frac{x^3}{l^2} + O(x^5)$$

Why prestression?

Usual spring

$$F_{restoring} \approx k\frac{x^3}{l^2} + O(x^5)$$

Doesn't lead to Sine-Gordon equation

(Proved analytically & numerically)

Prestressed spring

$$F_{restoring} \approx 2F_{prestression}\frac{x}{l} + k\frac{x^3}{l^2} + O(x^5)$$

Sine-Gordon eq.

$$F_{restoring} \approx 2F_{prestression}\frac{x}{l} + O(x^3)$$

Error in 3rd order found irrelevant by 4th order Runge-Kutta

Prestression is needed, elasticity is not important

Derivation of equation of motion

Using Principle of least action (Euler-Lagrange equation)

$$L = T - U_g - U_s \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} = 0$$

Term from Kinetic energy:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) = I\ddot{\theta}_i$$

Terms from Potential energy:

$$\frac{\partial L}{\partial \theta_i} = \frac{\partial}{\partial \theta_i}(-U_g) + \frac{\partial}{\partial \theta_i}(-U_s)$$

$$\frac{\partial}{\partial \theta_i}\left(-U_g\right) = -mgl_t \sin(\theta_i)$$

$$\frac{\partial}{\partial \theta_i}\left(-U_s\right) = \frac{Fl^2}{a}(\theta_{i-1} + \theta_{i+1} - 2\theta_i)$$

$$\ddot{\theta}_i = \frac{Fl^2}{Ia}\left(\theta_{i-1} + \theta_{i+1} - 2\theta_i\right) - \frac{mgl_t}{I}\sin(\theta_i)$$

Continuous limit: Derivation

Using Equation of motion

$$\ddot{\theta}_i = \frac{Fl^2}{Ia}\left(\theta_{i-1} + \theta_{i+1} - 2\theta_i\right) - \frac{mgl_t}{I}\sin(\theta_i)$$

Small difference in angle of adjacent pendula

$$\sin(x) \approx x$$

$$\left(\theta_{n+1} + \theta_{n-1} - 2\theta_i\right) \approx a^2 \frac{\partial^2 \theta}{\partial x^2} + \mathrm{O}\left(a^4 \frac{\partial^4 \theta}{\partial x^4}\right)$$

$$\frac{\partial^2 \theta}{\partial t^2} - c_0^2 \frac{\partial^2 \theta}{\partial x^2} + \omega_0^2 \sin\theta = 0$$

$$c_0^2 = \frac{Fl^2 a}{I}$$

$$\omega_0^2 = \frac{mgl_t}{I}$$

Sine-Gordon equation

known analytical solution [1]

[1] Physics of solitons. M. Peyrard, T. Dauxios, *Cambridge University Press (2010)*

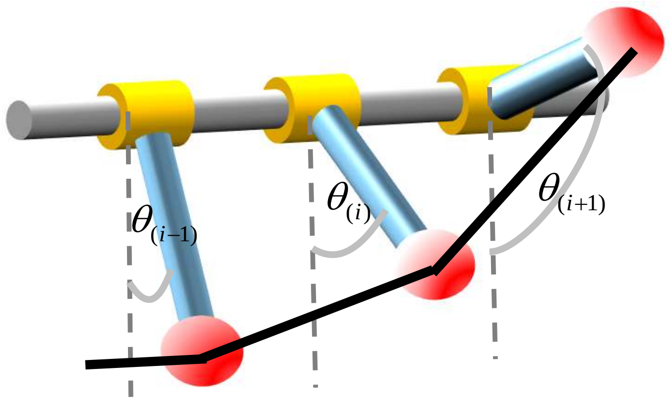
Chain of pendula: Equation of motion

Discrete system

$$\ddot{\theta}_i = \frac{Fl^2}{Ia}\left(\theta_{i-1} + \theta_{i+1} - 2\theta_i\right) - \frac{mgl_t}{I}\sin(\theta_i)$$

$$\ddot{\theta}_1 = -\frac{Fl^2}{Ia}\left(\theta_1 - \theta_2\right) - \frac{mgl_t}{I}\sin(\theta_1)$$

$$\ddot{\theta}_N = \frac{Fl^2}{Ia}\left(\theta_{N-1} - \theta_N\right) - \frac{mgl_t}{I}\sin(\theta_N)$$



Friction

Pendulum cannot move just rotate

$$\sum \vec{F} = 0 = \vec{F}_n + \vec{F}_g + \vec{F}_{spring}$$

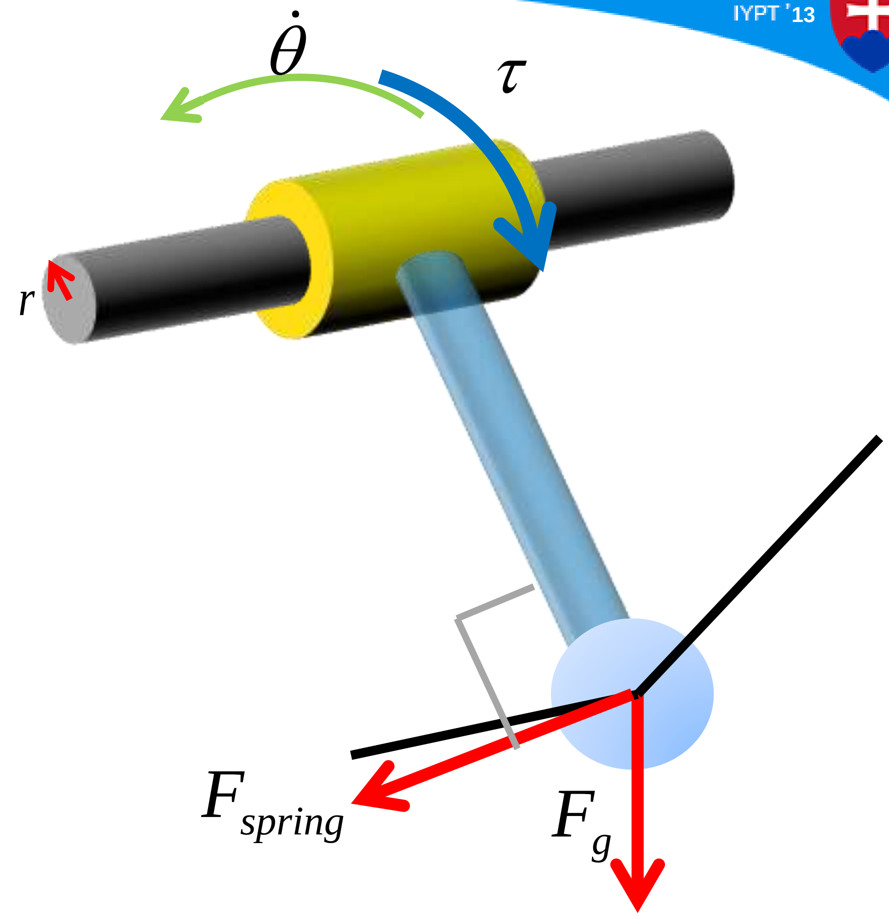
Normal force acting on the axis

$$\left|\vec{F}_N\right| = mg + \frac{Fl}{a}(\theta_{i+1} + \theta_{i-1} - 2\theta_i)$$

Causes torque of friction forces

$$F_f \le fF_N$$

$$\tau_f = rF_f$$

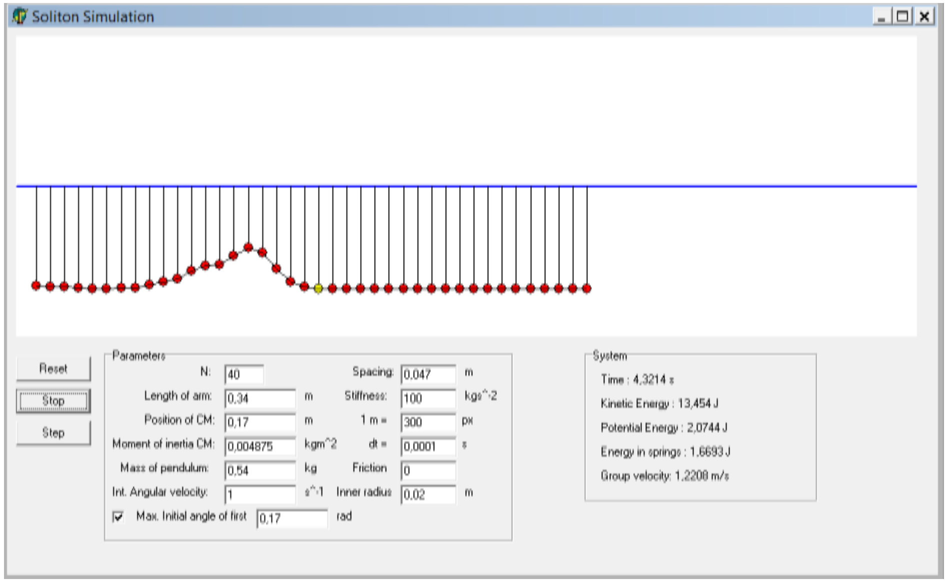


$$\ddot{\theta}_i = \frac{Fl^2}{Ia}\left(\theta_{i+1} + \theta_{i-1} - 2\theta_i\right) - \frac{mgl_t}{I}\sin(\theta_i) - \operatorname{sgn}(\dot{\theta})\frac{r}{I} f\left(mg + \frac{Fl}{Ia}\left(\theta_{i+1} + \theta_{i-1} - 2\theta_i\right)\right)$$

Simulation (4th Order Runge-Kutta)

<u>Using 4th order Runge-Kutta method:</u>

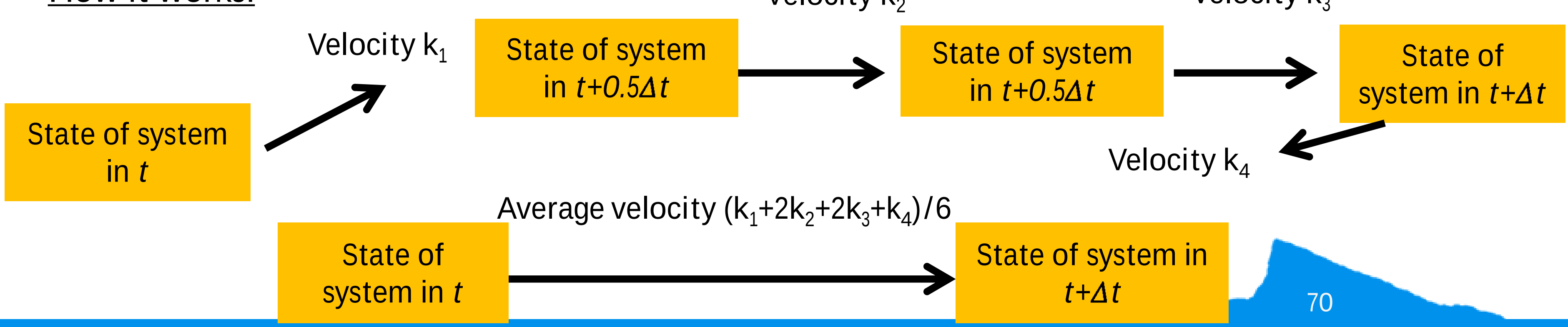
- More precise (With comparison to Euler method)
- Faster (Higher time step is sufficient to achieve the same accurancy)
- Problem is about nonlinearity
  1st order method was unstable



<u>Simulating equation of motion for every pendulum</u>

$$\ddot{\theta}_i = \frac{kl^2}{I}\left(\sin(\theta_{i-1}-\theta_i)-\sin(\theta_i-\theta_{i+1})\right)-\frac{mgl_t}{I}\sin(\theta_i)-\mathrm{sgn}(\dot{\theta})\frac{r}{I}f\left(mg+\frac{kl}{I}\left(\sin(\theta_{i-1}-\theta_i)-\sin(\theta_i-\theta_{i+1})\right)\right)$$

<u>How it works:</u>



Dispersion: Derivation

$$\frac{\partial^2\theta}{\partial t^2} - c_0^2 \frac{\partial^2\theta}{\partial x^2} + \omega_0^2 \sin\theta = 0 \quad \xrightarrow{\sin\theta \approx \theta} \quad \frac{\partial^2\theta}{\partial t^2} - c_0^2 \frac{\partial^2\theta}{\partial x^2} + \omega_0^2 \theta = 0$$

Assuming wave in form: $\theta_{(x,t)} = Ae^{i(\omega t + kx)}$

Taking into the limit of Sine-Gordon equation:

$$-\omega^2 Ae^{i(\omega t+kx)} + c_0^2 k^2 Ae^{i(\omega t+kx)} + \omega_0^2 Ae^{i(\omega t+kx)} = 0$$

$$\omega = \pm\sqrt{\omega_0^2 + c_0^2 k^2}$$

$$v_{group} \equiv \frac{\partial\omega}{\partial k} = \frac{c_0^2 k}{\sqrt{\omega_0^2 + c_0^2 k^2}} = \frac{2\pi c_0^2}{\sqrt{\lambda^2\omega_0^2 + 4\pi^2 c_0^2}} = \frac{2\pi F l^2 a}{\sqrt{\lambda^2 m g l_t I + I F l^2 a}}$$

Derivation: Group velocity of Soliton

- Angular velocity at maximum point

$$\omega_{initial} = \frac{\partial \theta_{(x,t)}}{\partial t}\Big|_{x=0,t=0}$$

$$\omega^2_{initial} = \left(\frac{\partial \theta_{(x,t)}}{\partial t}\right)^2 = v^2\left(\frac{\partial \theta_{(x,t)}}{\partial z}\right)^2 \qquad z = x - vt \qquad \theta_{(x,t)} = 4ArcTan\left(Exp\left(\frac{\omega_0(x-vt)}{c_0\sqrt{1-\frac{v^2}{c_0^2}}}\right)\right)$$

$$\left(\frac{\partial \theta_{(x,t)}}{\partial z}\right)^2 = 4\frac{\omega_0^2}{c_0^2\left(1-\frac{v^2}{c_0^2}\right)} Sech^2\left(\frac{\omega_0(x-vt)}{c_0\sqrt{1-\frac{v^2}{c_0^2}}}\right) \qquad Sech(0) = 1$$

$$|v| = \sqrt{\frac{\omega^2_{initial}}{4\omega_0^2 + \omega^2_{initial}}}c_0$$

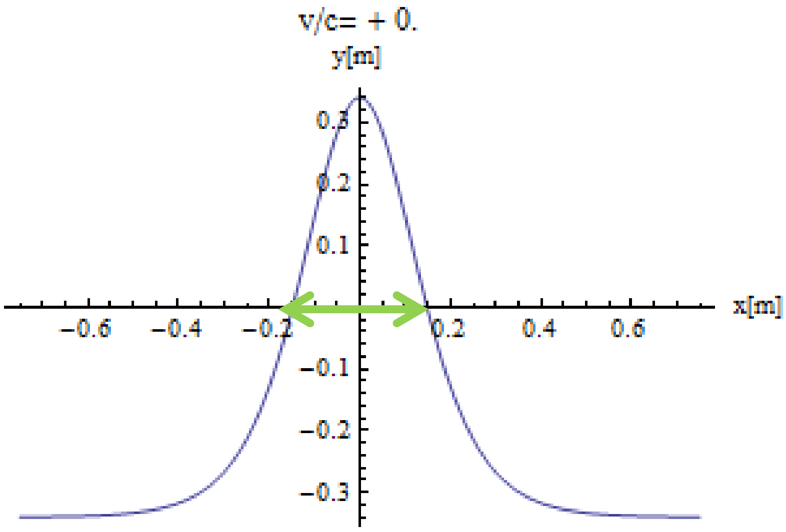
Analogy

- Typical length of soliton decreases with increasing group velocity

$$\theta_{(x,t)} = 4ArcTan\left(Exp\left(\frac{\omega_0 (x - vt)}{c_0\sqrt{1-\frac{v^2}{c_0^2}}}\right)\right) \rightarrow \frac{1}{L}$$

- Analogy to Special Theory of Relativity – Contraction of length

$$L = \underbrace{\frac{c_0}{\omega_0}}_{L_0}\sqrt{1-\frac{v^2}{c_0^2}}$$

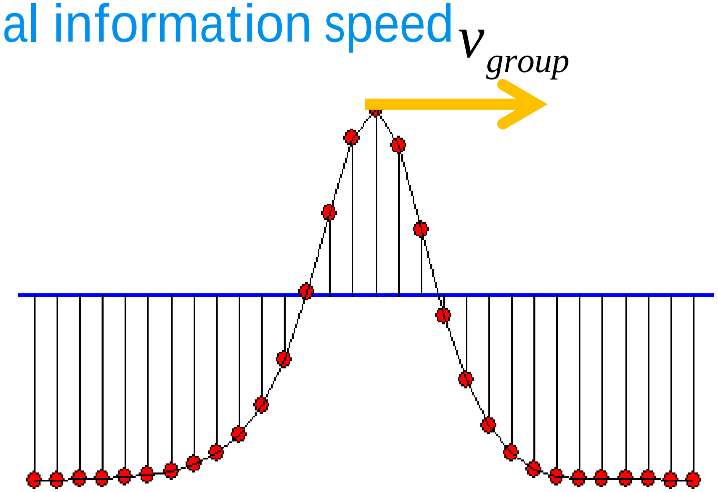


Two theoretical limits

- Limit of maximal group velocity - maximal information speed

$$\omega_{initial} \to \infty \qquad v_{group} = c_0$$

Actually never in our world (Speed of light)



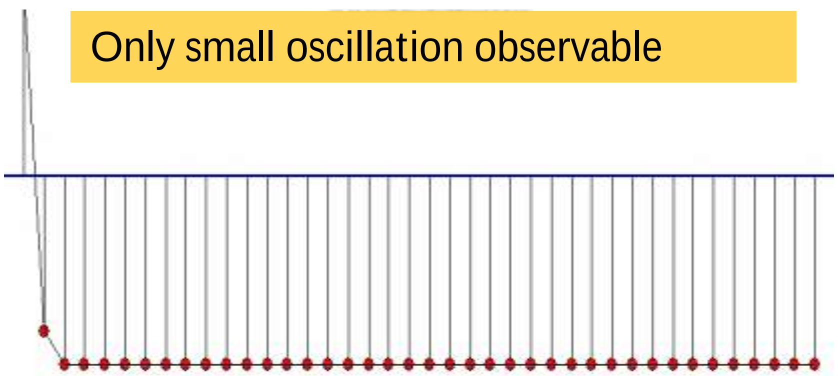
- **"Invisibility"** – Consequence of contraction in discrete system

If the "length of soliton" is much smaller than a

$$\omega_{initial} >> \sqrt{\frac{\frac{Fl^2}{a} - mgl_t}{I}}$$

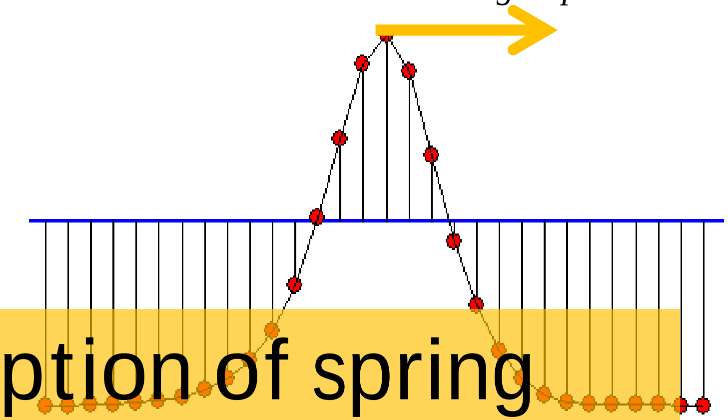
(Derivation in appendices)

$$\omega_{initial} \approx 100 \ rads^{-1}$$



Two theoretical limits

- Limit of maximal group velocity - maximal information speed v_{group}



$$\omega_{initial} \to \infty \qquad v_{group} = c_0$$

Actually never in our world (Speed of light)

Experimental limit – Disruption of spring

- **"Invisibility"** – Consequence of contraction in discrete system

If the "length of soliton" is much smaller than a

$$\omega_{initial} >> \sqrt{\frac{\frac{Fl^2}{a} - mgl_t}{I}}$$

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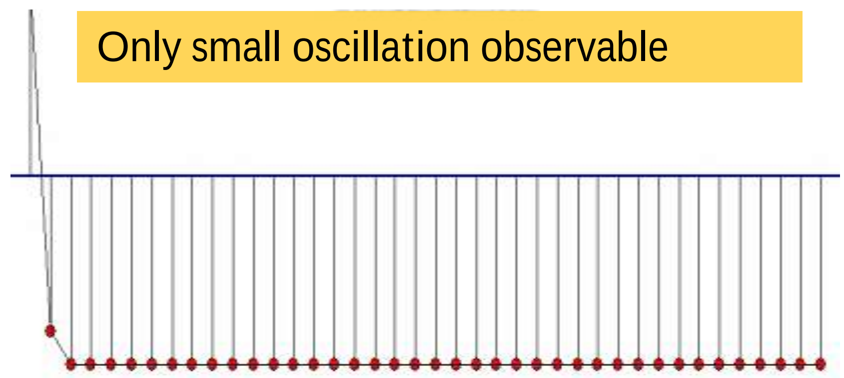

"Invisibility"

If the "length of soliton" is much smaller than a

$$L << a \Rightarrow \frac{c_0}{\omega_0}\sqrt{1-\frac{v^2}{c_0^2}} << a \quad + \quad |v| = \sqrt{\frac{\omega_{initial}^2}{4\omega_0^2+\omega_{initial}^2}}c_0$$

$$\omega_0^2 = \frac{mgl_t}{I} \qquad c_0^2 = \frac{Fl^2a}{I}$$

$$\omega_{initial} >> \sqrt{\frac{\frac{F}{a}l^2 - mgl_t}{I}}$$

What if $\frac{Fl^2}{a} < mgl_t$?

- Then Continuous approach cannot be used
- It requires $L_0 = \frac{c_0}{\omega_0} >> a$ that means $\frac{Fl^2}{a} >> mgl_t$

Energy in springs >> Gravitational energy

Ultimate answer to all: ”What if?”

		Small deflection	360° deflection
Without gravity	Without friction	Stable profile	Large wave moving along chain
Without gravity	Friction	Dissipation of Stable profile	Dissipation of Large wave moving along chain
With gravity	Without friction	Dispersion of initial profile	Soliton
With gravity	Friction	Dispersion & Dissipation of initial profile	Dissipation of Soliton