IYPT – AUSTRALIA
QUESTION 17: CRAZY SUITCASE

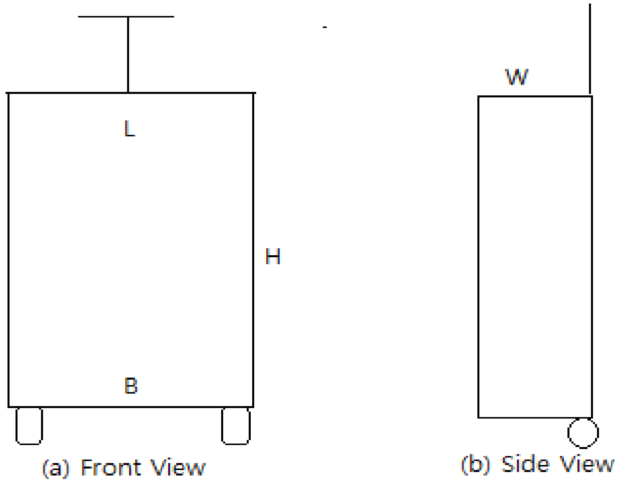
Reporter: Jeong Han Song

PROBLEM

When one pulls along a **two wheeled suitcase,** it can under certain circumstances **wobble so strongly** from side to side that it can **turn over.** Investigate this phenomenon. Can one **suppress** or **intensify** the effect by **varied packing** of the luggage?

TERMS & DEFINITIONS

When one pulls along a **two wheeled suitcase**, it can under certain circumstances **wobble so strongly** from **side to side** that it can **turn over.** Investigate this phenomenon. Can one suppress or intensify the effect by **varied packing** of the luggage?



$$L = length = 40\ cm$$
$$H = height = 50cm$$
$$B = distance\ between\ two\ wheels = 39cm$$
$$W = width = 20\ cm$$

$$\tau = Fd$$

Detachment of both wheels

Different Positions of Centre of mass

FLOW CHART

Qualitative Analysis

- Stages of motion of the wobbling suitcase
- Condition of overturn

Quantitative Analysis

- Net Torque Equation
- Human Effect (Literature Review)
- Graphical Simulation (Theoretical Prediction)

Experiment

- Measurement of angular displacement
- Computational Analysis
- Angular Acceleration (Derivation of force)

Additional circumstances

- Walking frequency
- Coefficient of Restitution
- Different mass density
- Angle of tilt

Relevant Condition for Overturn

QUALITATIVE ANALYSIS

- Cases for different effective height
- Five different phases

STAGE 1:
INITIAL CONDITION

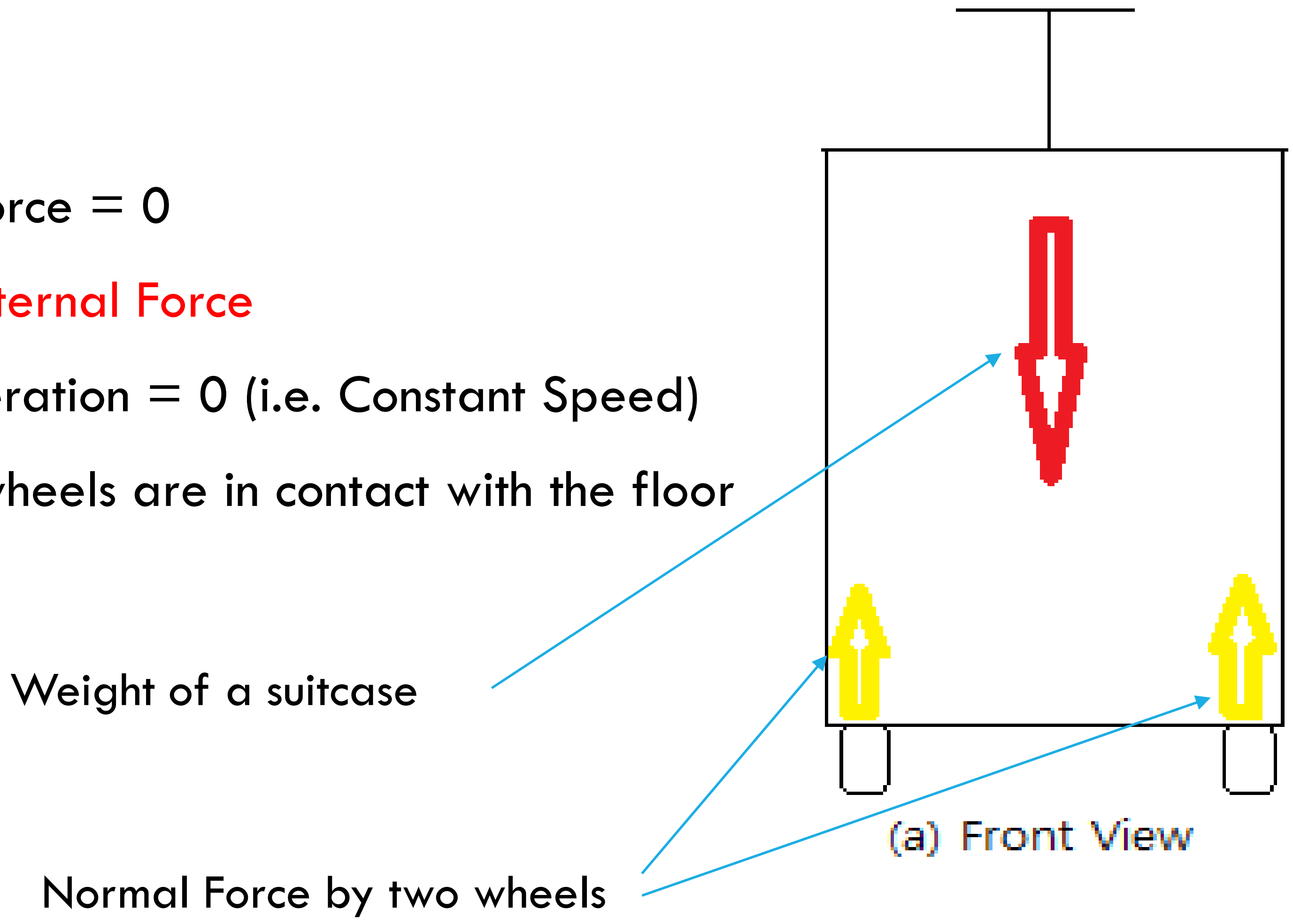
Titled

Net Force = 0

No External Force

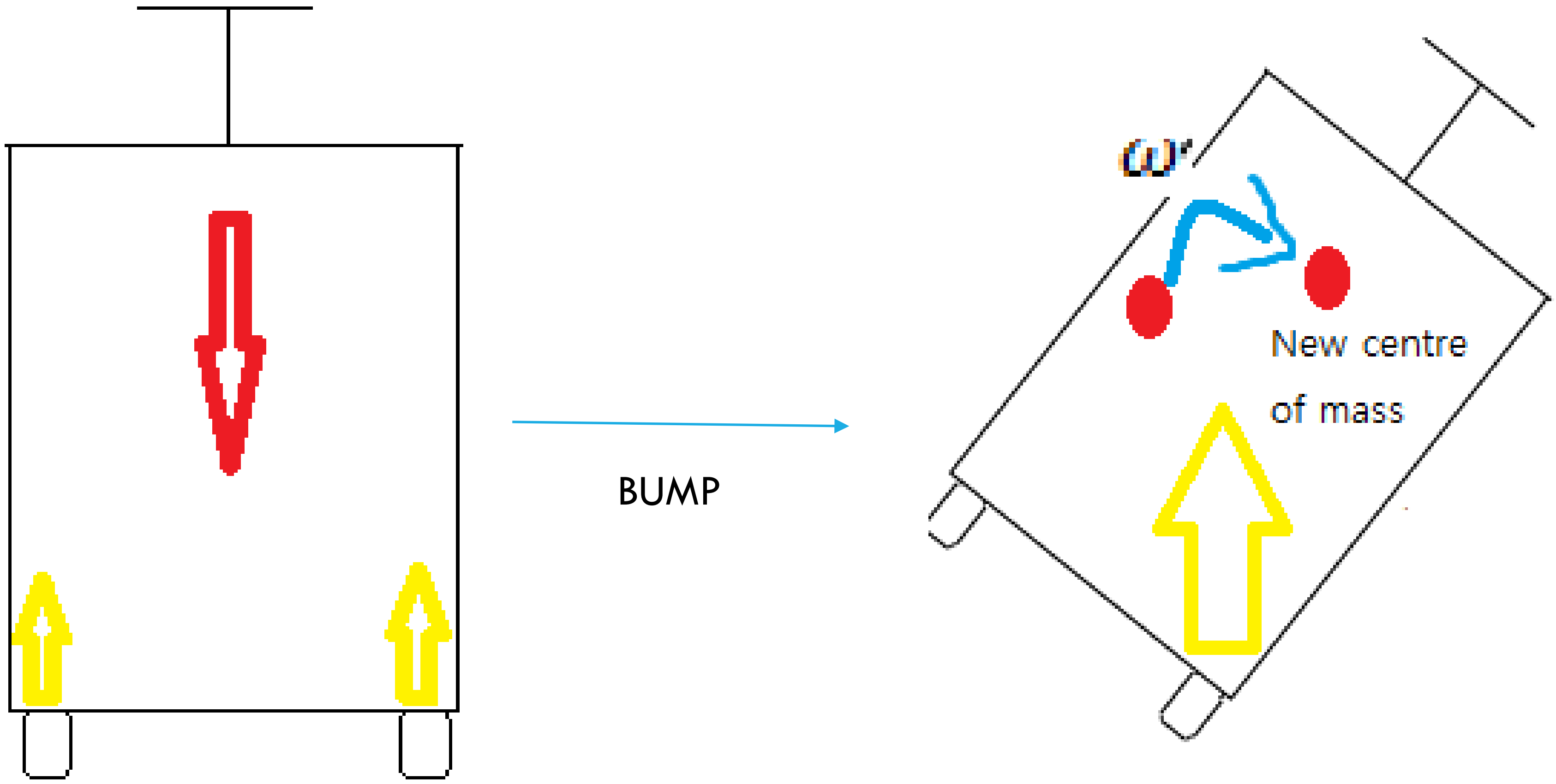
Acceleration = 0 (i.e. Constant Speed)

Both wheels are in contact with the floor



(a) Front View

STAGE 2:
EFFECT OF INITIAL DISTURBANCE

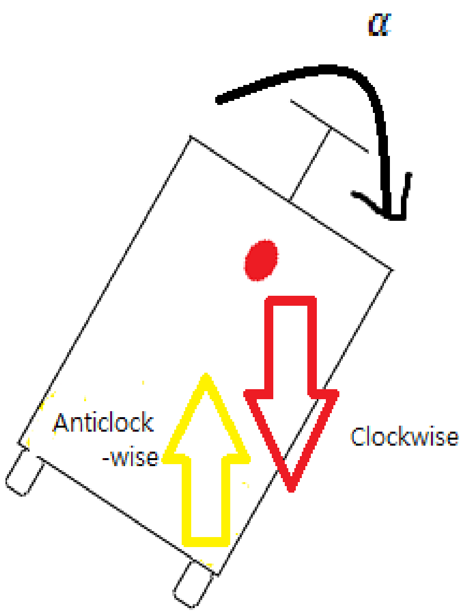


(a) Front View

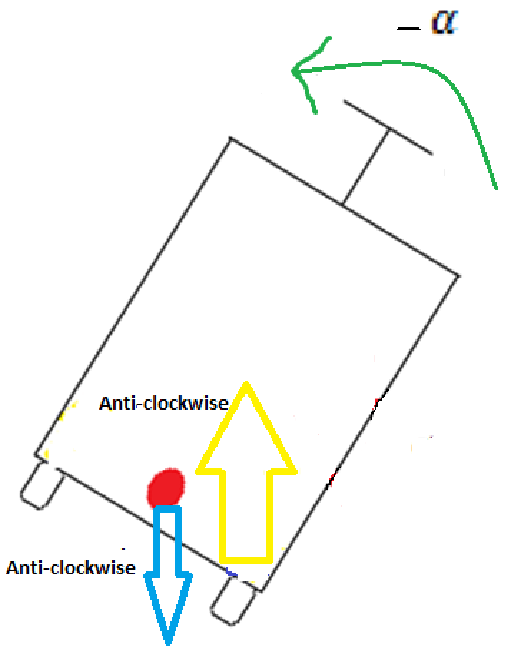
(a) Front View

STAGE 2:
EFFECT OF INITIAL DISTURBANCE (TORQUE)

- Different effect depending on the position of weight
- Case 1: High weight
    - Centre of mass now causes rotation about the supporting wheel
    - Torque created by the center of mass
    - Angular acceleration is present
- Case 2: Low weight
    - Low centre of mass acts as a restoring force that opposes the original torque created by the initial disturbance
    - Diminishes the wobble effect



(a) Front View



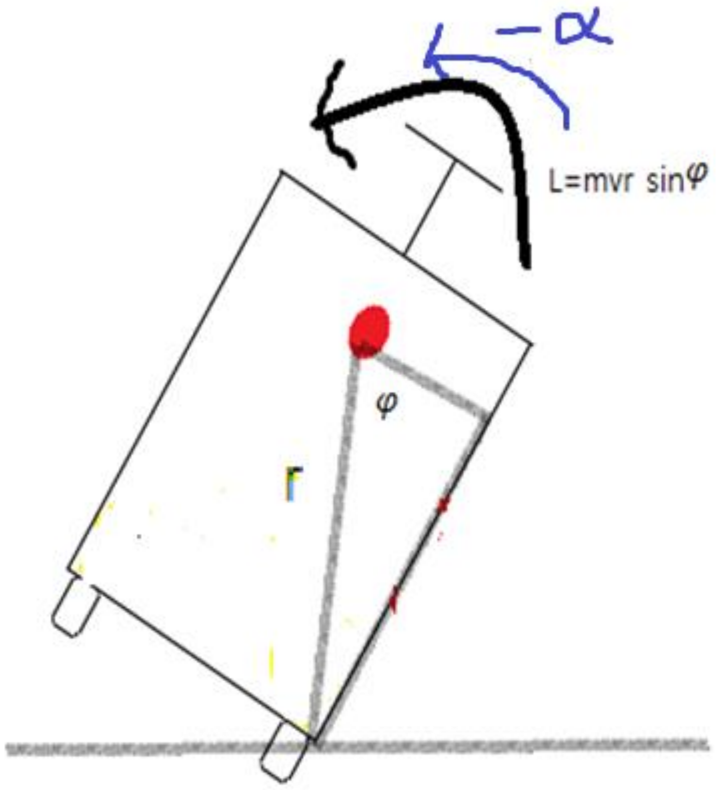
(a) Front View

STAGE 3: HUMAN RESPONSE

Human exerts a periodic force that opposes the disturbance created by the weight.

Opposite direction to the torque created by the weight

Effect of inertia

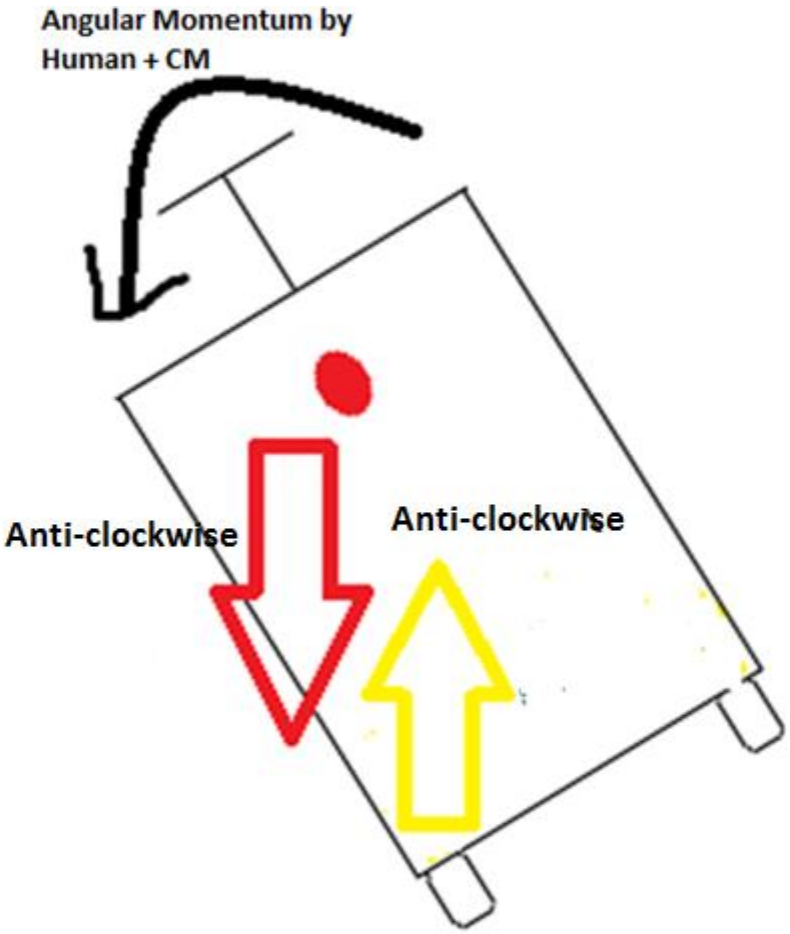


(a) Front View

STAGE 4:
POINT OF INSTABILITY

The restoring force overshoots

Overcompensation of restoring force would lead to the rather sharp increase in the oscillation.

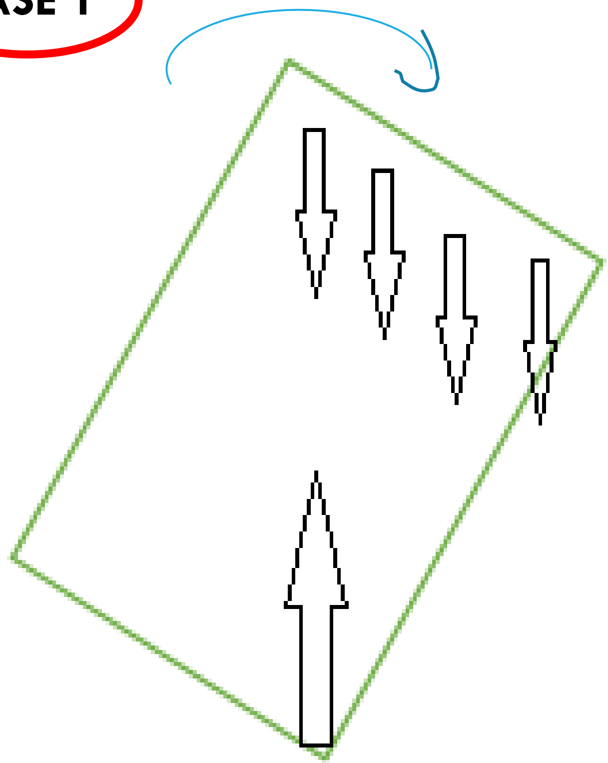


STAGE 5:
REPETITION AND AMPLIFICATION

When a suitcase meets a critical amplitude that exceeds what human can counteract, it overturns.

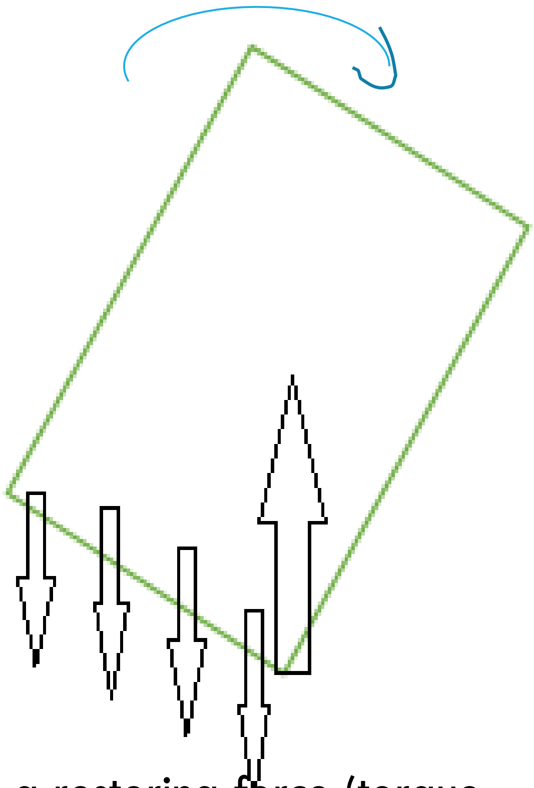
QUALITATIVE MODEL – SUMMARY

CASE 1



Intensifies the initial disturbance (torque is applied in the same direction as disturbance)

CASE 2

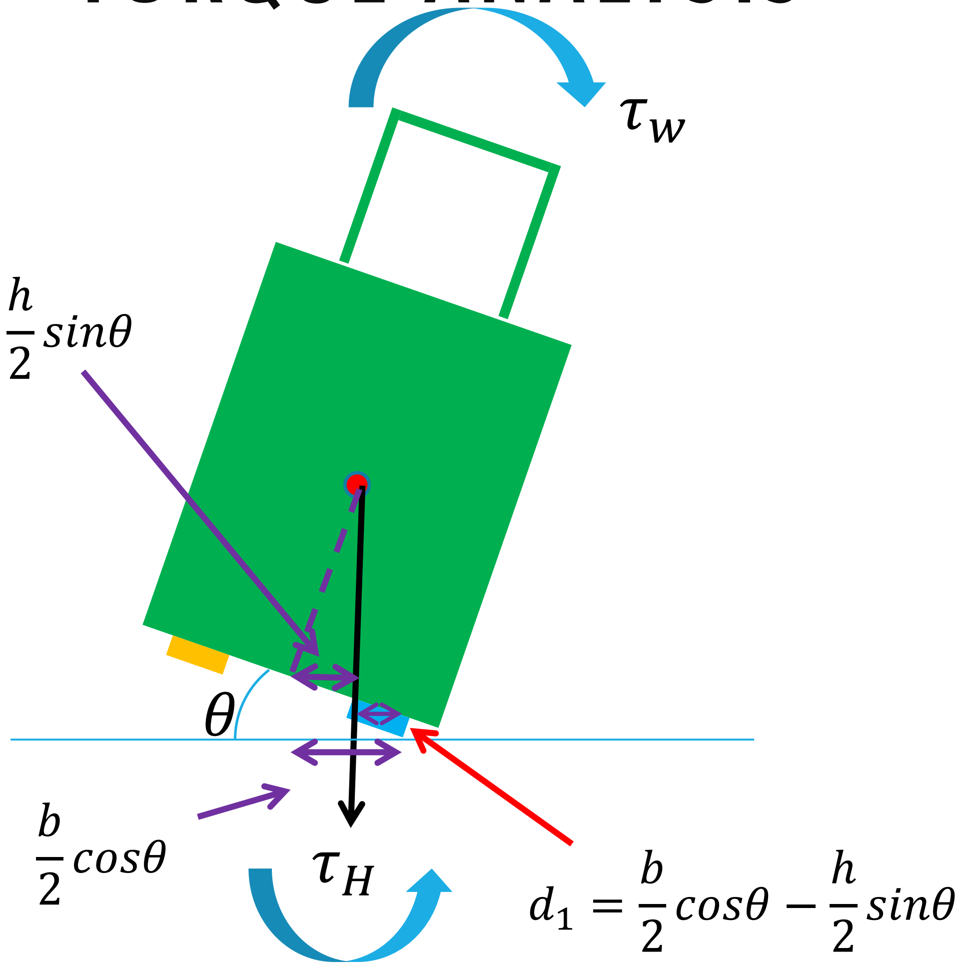


Acts as a restoring force (torque is applied in the opposite direction as disturbance)

QUANTITATIVE ANALYSIS

- Net Torque Equation
- Literature Review: Suherman's model
- Theoretical Prediction (Graphical Simulation)

TORQUE ANALYSIS



Two Types of torque:

1) Torque by Weight

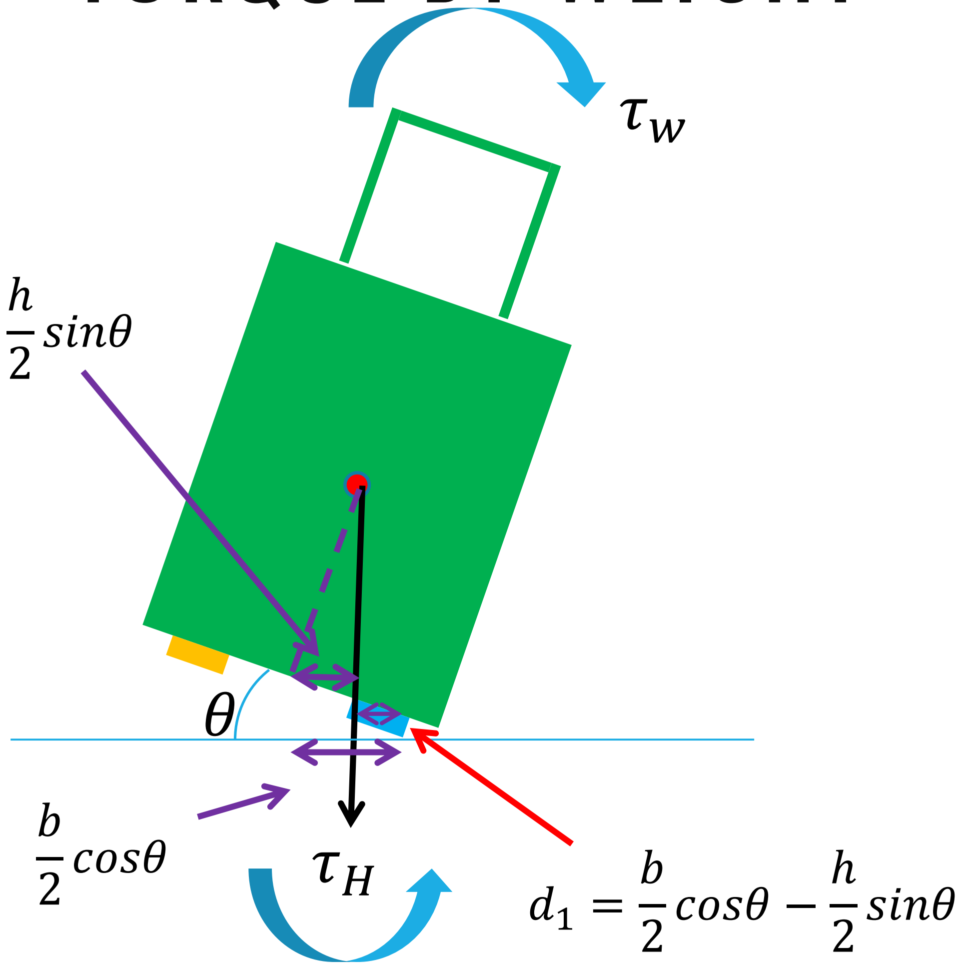
2) Torque by Human

Net Torque Equation:

$$I\frac{d\theta}{d^2t} = \tau_H - \tau_w$$

(Opposite Direction)

TORQUE BY WEIGHT



From basic torque equation:

$$\tau = F \times d_1$$

Force is equivalent to the weight

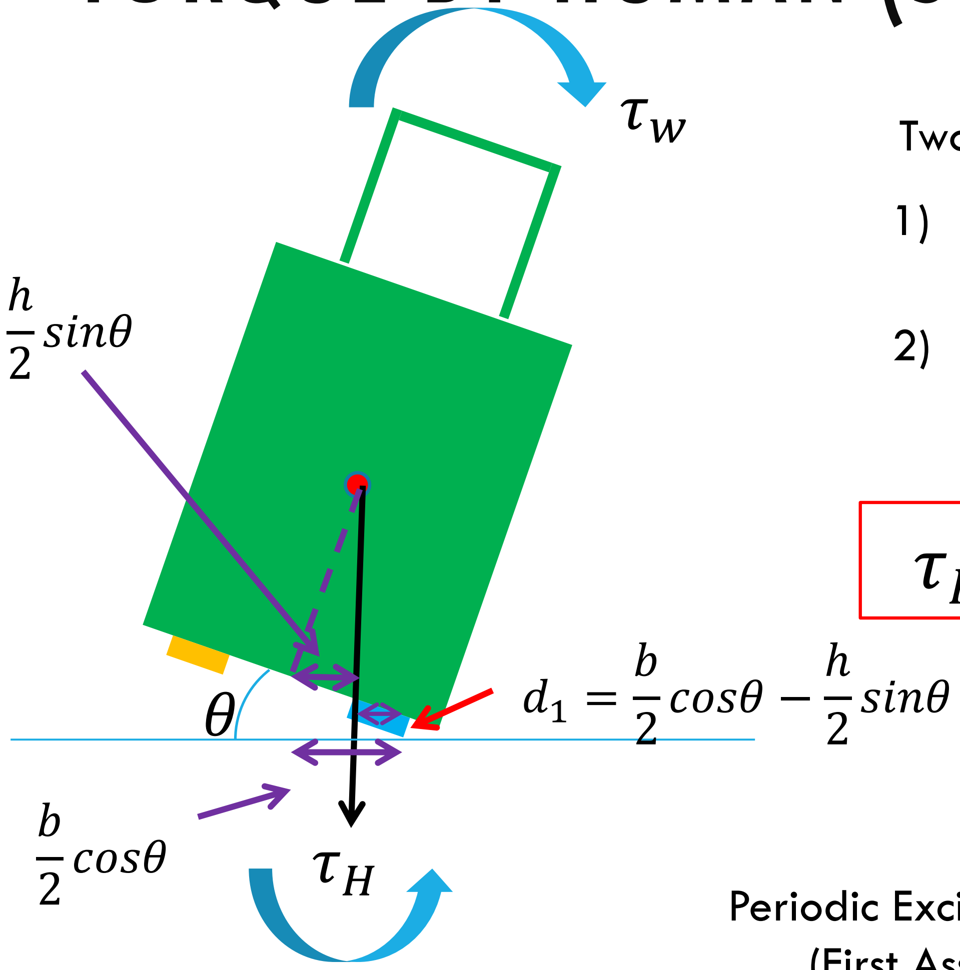
$$F = mg$$

Shortest distance is:

$$\Rightarrow \frac{b}{2}\cos\theta - \frac{h}{2}\sin\theta$$

$$\therefore \tau_W = mg(\frac{b}{2}\cos\theta - \frac{h}{2}\sin\theta) \quad \ldots (1)$$

TORQUE BY HUMAN (SUHERMAN'S MODEL)



Two Fundamental Assumptions:

1) Puller's walking motion induces a periodic moment on the handle of suitcase
2) Puller exerts additional restoring moment to suppress the wobble proportional to rocking angle

$$\tau_H = q_\theta \sin(\omega t) - k\theta \quad \ldots (2)$$

Periodic Excitation Torque (First Assumption)

Restoring Torque (Second Assumption)

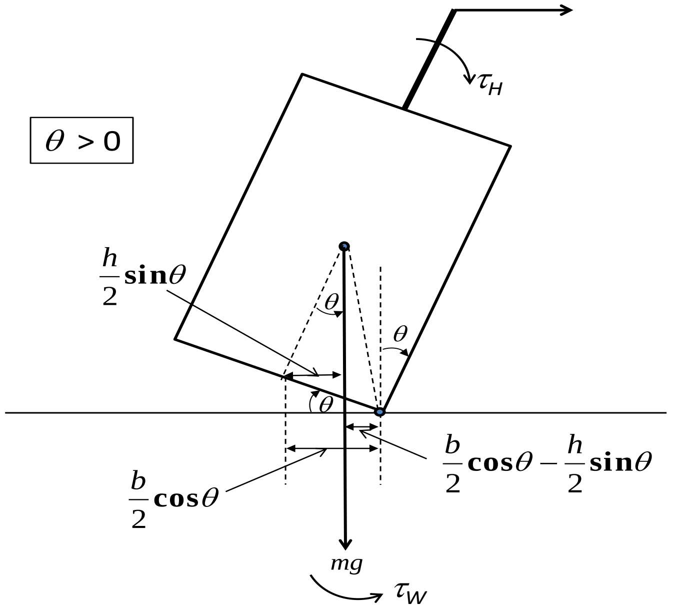
θ: angle of rocking

q_θ : amplitude of excitation torque

ω: walking frequency

k: constant for restoring torque

NET TORQUE EQUATION



Combining Equation 1 & 2:

$$I\frac{d\theta}{d^2t} = \tau_H - \tau_w$$

$$I\frac{d\theta}{d^2t} = q_0\sin(\omega t) - k\theta - mg\left(\frac{b}{2}\cos\theta - \frac{h}{2}\sin\theta\right) \quad \ldots\theta > 0$$

Or

$$I\frac{d\theta}{d^2t} = q_0\sin(\omega t) - k\theta + mg\left(\frac{b}{2}\cos\theta + \frac{h}{2}\sin\theta\right) \quad \ldots\theta < 0$$

FINAL NET TORQUE EQUATION

Define: $S = +1, if\ \theta > 0$ $\quad S = -1, if\ \theta < 0$

→ SIMPLE

Final Equation:

→ ACCURATE

$$I\frac{d^2\theta}{dt^2} = q_0 \sin(\omega t) - k\theta - Smg\frac{b}{2}\cos\theta + mg\frac{h}{2}\sin\theta$$

b: bottom length of a suitcase

h: effective height

θ: angle of rocking

q_θ : amplitude of excitation torque

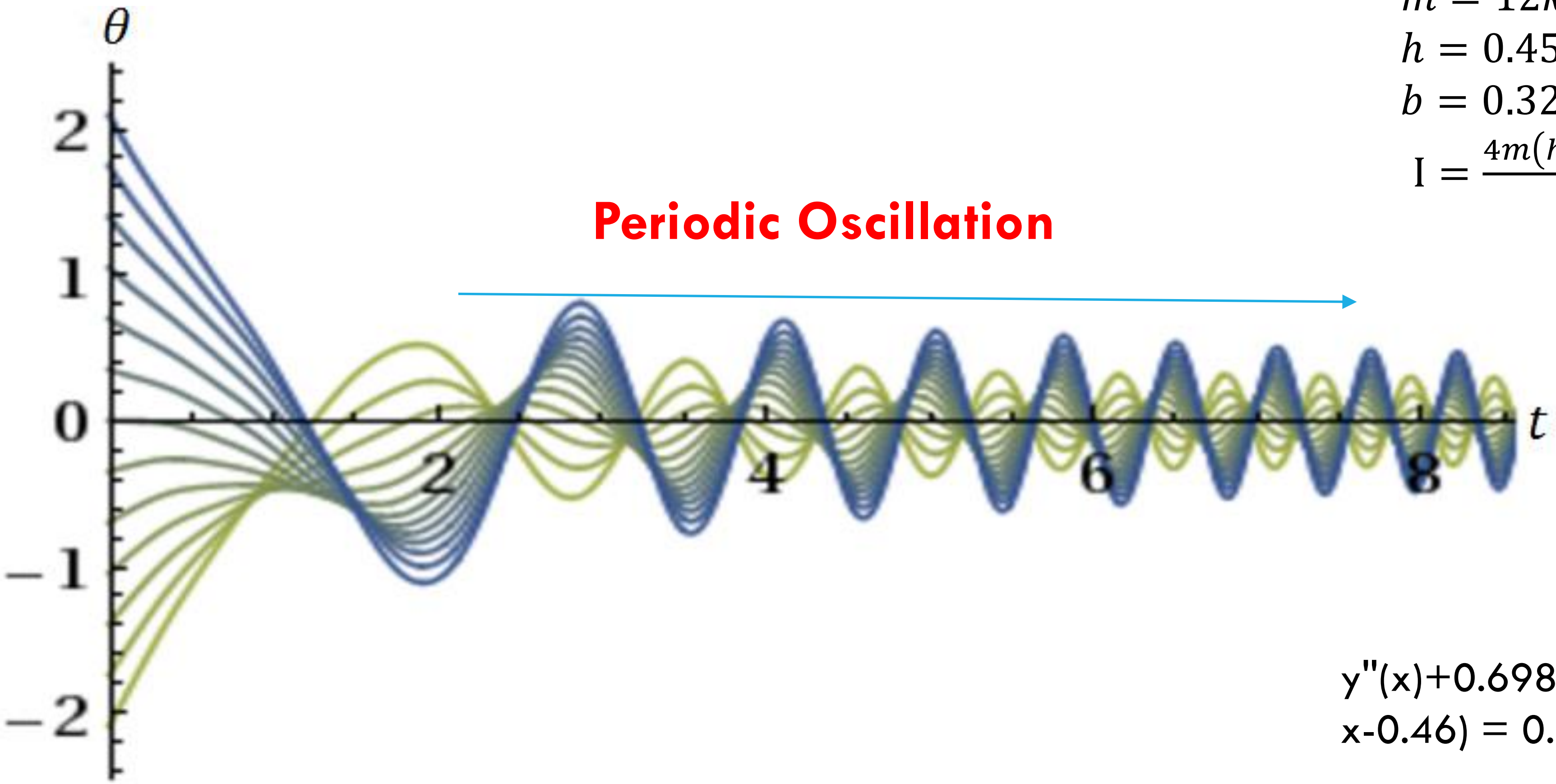
ω: walking frequency

k: constant for restoring torque

As effective height increases, net torque increases!

THEORETICAL PREDICTION – WOBBLE

Sample Graph from our computer simulation (wolfram alpha) using values below:



$m = 12kg$

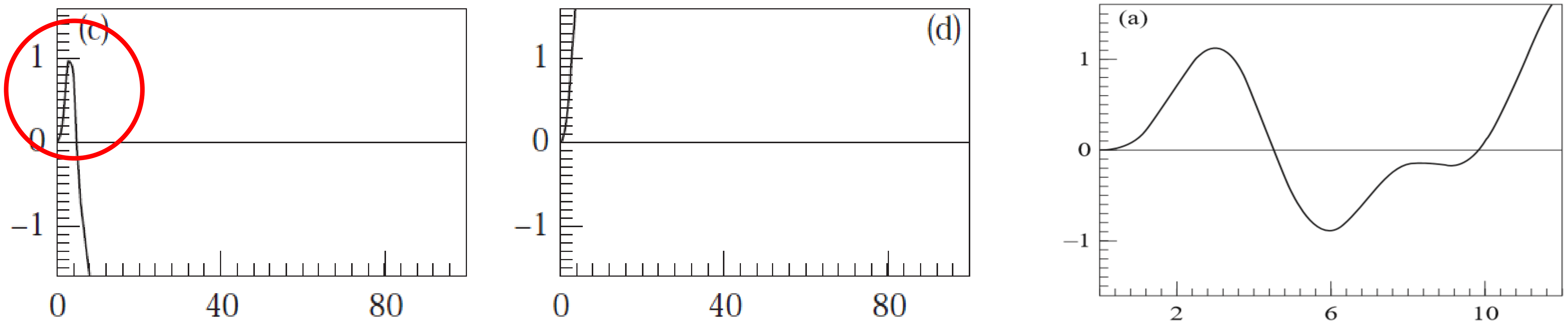
$h = 0.458m$

$b = 0.320m$

$$\mathrm{I} = \frac{4m(h^2+b^2)}{3} = 4.996\ kgm^2$$

y''(x)+0.69869 cos(y(x))-sin(y(x))+y x (2.3232 x-0.46) = 0.6 sin(4.9946×2.3232 x+0.426)

LITERATURE REVIEW: SUHERMAN'S RESULTS OVERTURN OF A SUITCASE



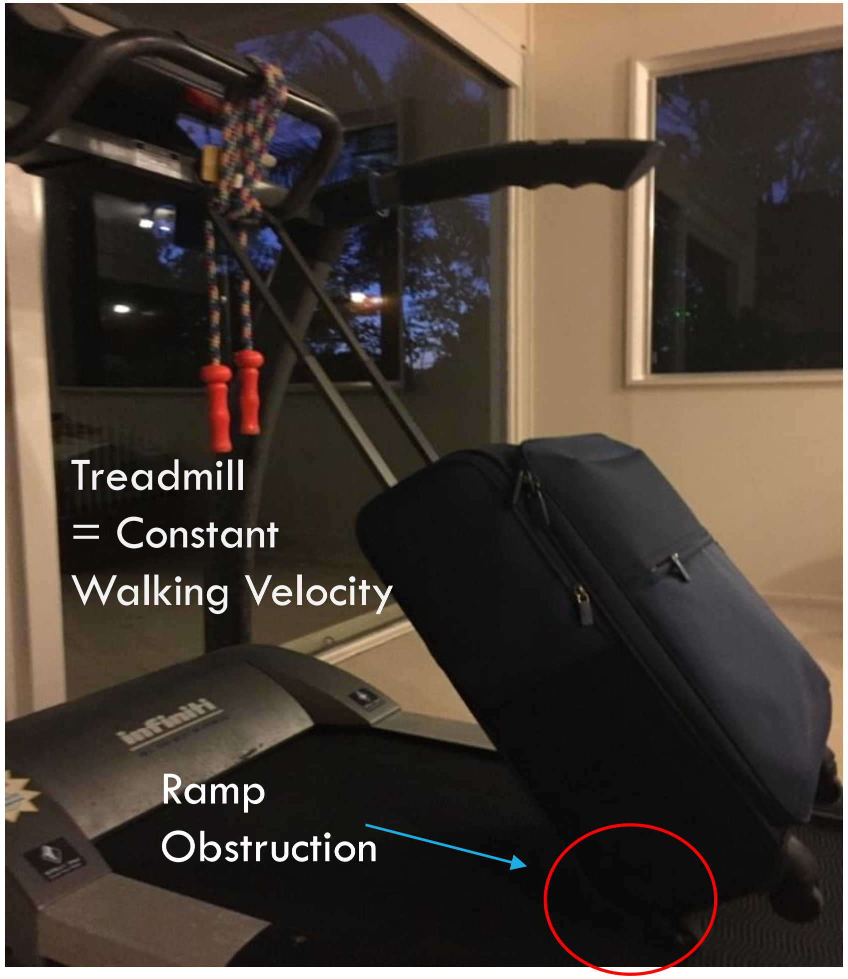
1. Sharp change in angular displacement
2. No wobble during the overturn

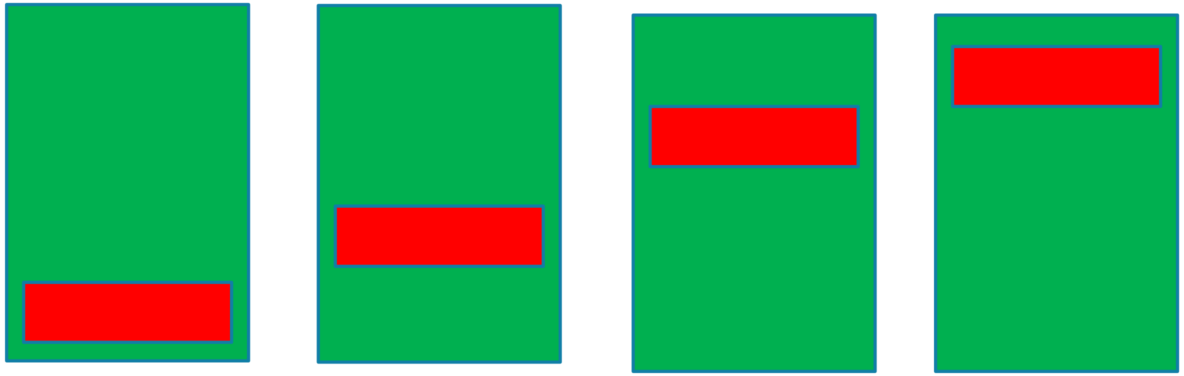
ROCKING ANGLE v TIME
OVERTURN ($\theta > \frac{\pi}{2}$)

- Experimental Setup
- Angular displacement vs. time
- Computational Analysis

EXPERIMENT

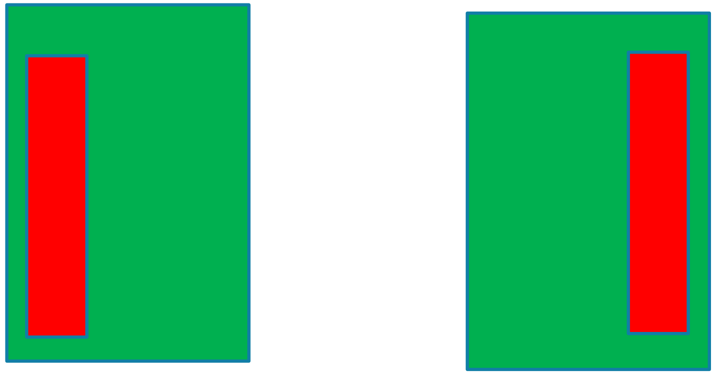
EXPERIMENTAL SETUP





SET A

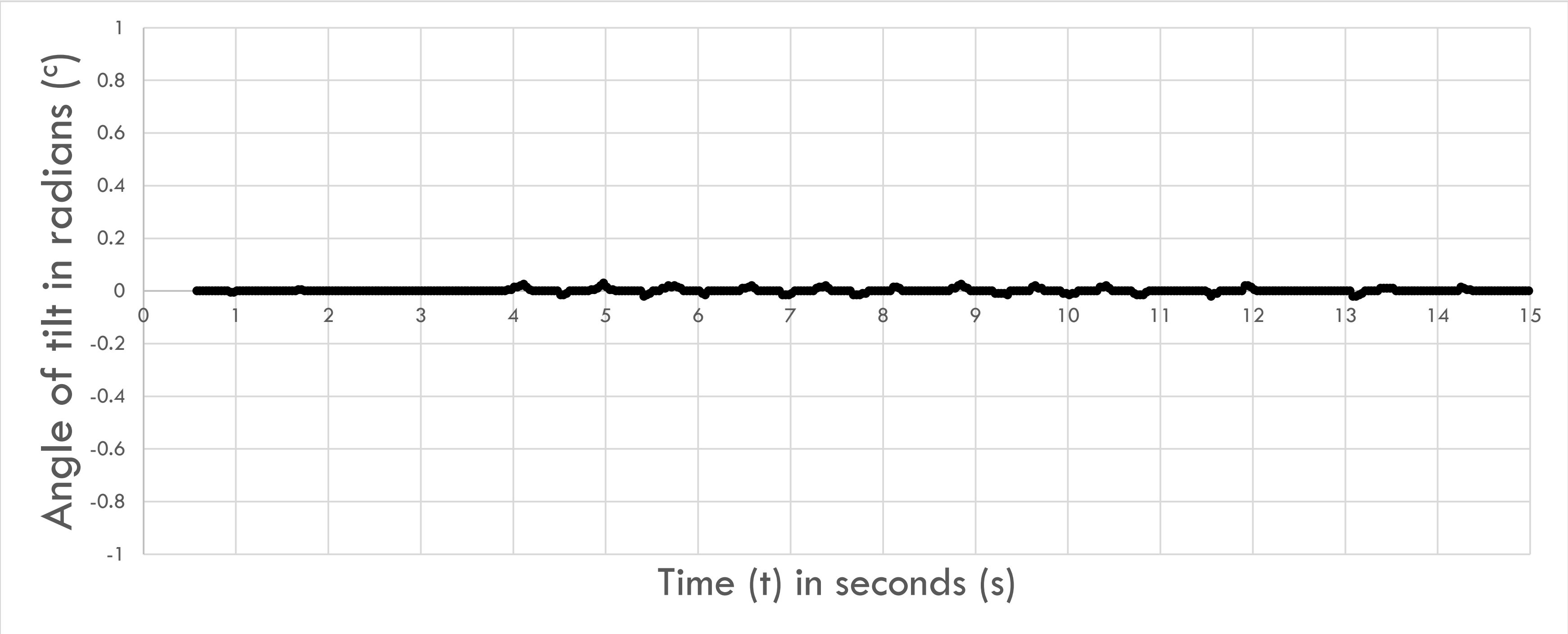
Height varied: 0.08m, 0.16m, 0.24m, 0.32m, 0.4m



SET B

Trackers Video Analysis

RESULT 1: 0.08M EFFECTIVE HEIGHT OF CM



- Very minimal angular displacement
- Inaccurate measurement (interval: 0.1s)
- Hard to observe trend or data

→ **Fast Fourier Transform**

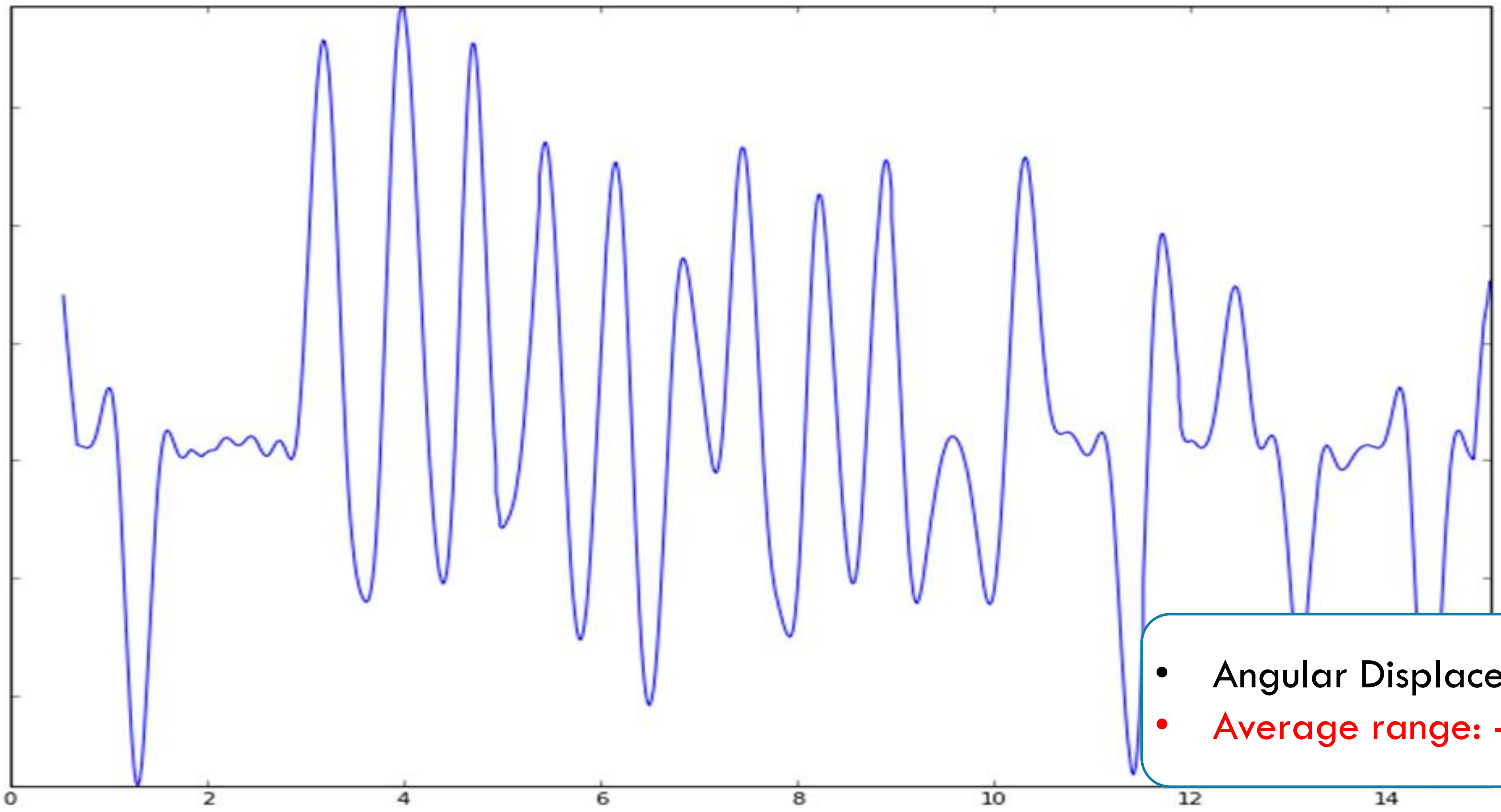
SIMULATING THE EQUATION

In [2]:

```
import numpy

data = numpy.genfromtxt(txt_file_name)
y_data = numpy.genfromtxt(txt_file_name, usecols=1)
Coef = numpy.fft.fft(y_data)
Freq = numpy.fft.fftfreq(len(y_data))
import matplotlib.pyplot as plt
Abs_Coef = numpy.abs(Coef)
%matplotlib inline
#plt.plot(Freq[:int(len(Freq)/2)], Abs_Coef[:int(len(
Pop_Coef = Coef[:equation_length]
Pop_Freq = Freq[:equation_length]
def graph(formula, x_range):
    x = numpy.array(x_range)
    y = eval(formula)
    plt.plot(x, y)
    plt.show()

Pop_Freq_complex = Pop_Freq.view(dtype=numpy.complex1

equation = str(0)
for i in range(0, len(Pop_Coef)):
    equation = equation + '+' + str(Pop_Coef[i]) + '*numpy.exp(' + str(Pop_Freq[i]) + 'j*x)'

#graph(equation, range(0, graph_length))
print('')
print('The equation of the original graph:')
print('')
print (equation)
```

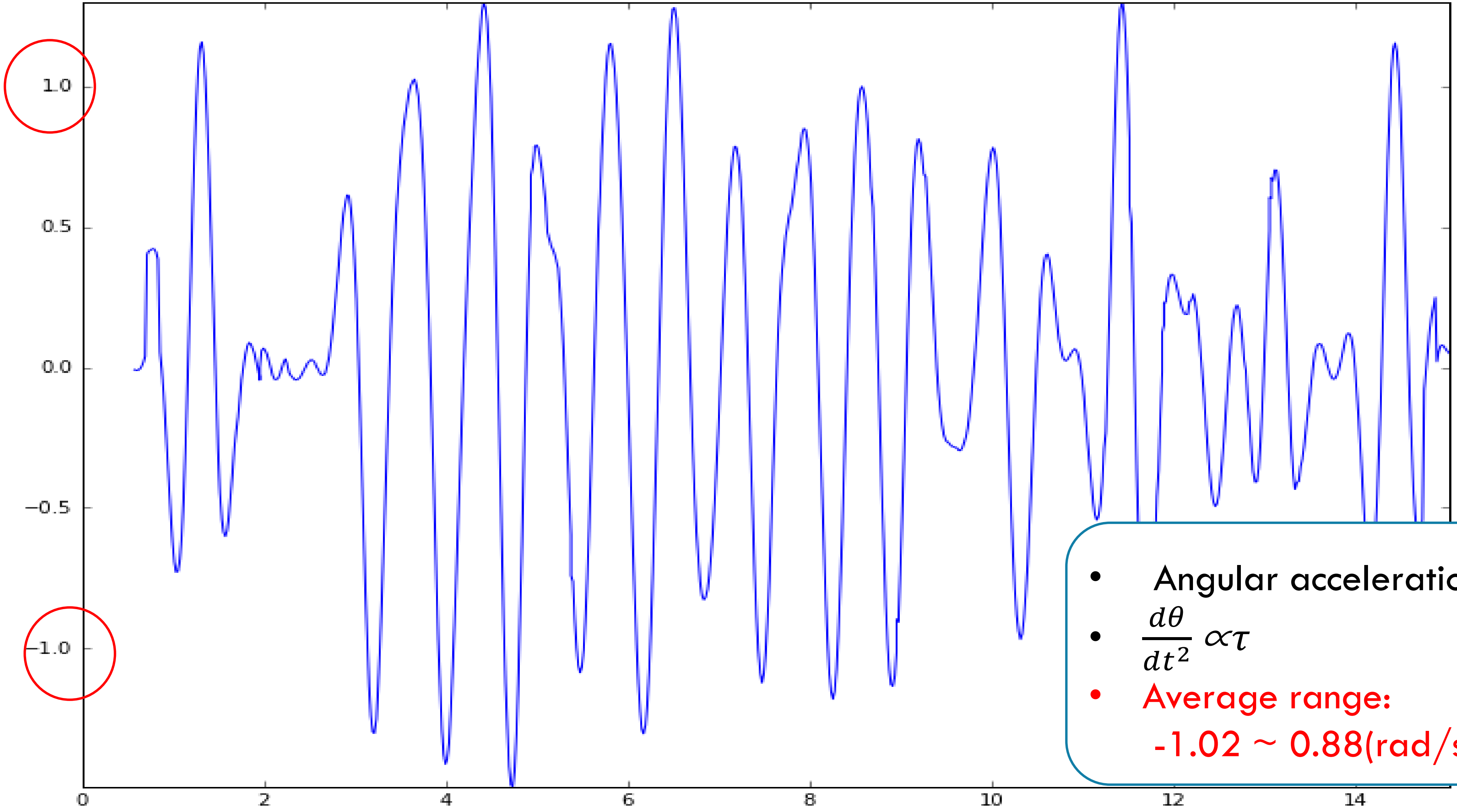
```
The equation of the original graph:

0+(1.67571644042+0j)*numpy.exp(0.0j*x)+(-0.39338130797-0.228859443672j)*numpy.exp(0.00135869565217j*x)+(-0.2407771180
86+0.294098934776j)*numpy.exp(0.00271739130435j*x)+(0.160951518353+0.292344341302j)*numpy.exp(0.00407608695652j*x)+
(0.0793870251769-0.0343400073869j)*numpy.exp(0.0054347826087j*x)+(0.128171884923+0.114165718316j)*numpy.exp(0.0067934
7826087j*x)+(-0.0789217710141+0.0506876519715j)*numpy.exp(0.00815217391304j*x)+(-0.313683719822+0.177882753422j)*nump
y.exp(0.00951086956522j*x)+(-0.405657156874-0.341040410623j)*numpy.exp(0.0108695652174j*x)+(0.16983120829-0.27419970
4291j)*numpy.exp(0.0122282608696j*x)+(-0.212322076372-0.0533675435594j)*numpy.exp(0.013586956521j*x)+(-0.1627146746l
5+0.0991007955451j)*numpy.exp(0.0149456521739j*x)+(-0.237838649698+0.49273818019j)*numpy.exp(0.0163043478261j*x)+(0.2
43829604558-0.0486385296346j)*numpy.exp(0.0176630434783j*x)+(-0.122470662683-0.0720897431264j)*numpy.exp(0.0190217391
304j*x)+(-0.353615922653-0.0570099119301j)*numpy.exp(0.0203804347826j*x)+(0.086710240439+0.322257867439j)*numpy.exp
(0.0217391304348j*x)+(0.369227221677-0.553337430946j)*numpy.exp(0.023097826087j*x)+(-0.690435890344-1.1958190939j)*nu
mpy.exp(0.0244565217391j*x)+(0.83962513333+0.98398930061j)*numpy.exp(0.0258152173913j*x)+(-0.374196414629-0.172350054
05j)*numpy.exp(0.0271739130435j*x)+(-0.102805844489+0.122797370408j)*numpy.exp(0.0285326086957j*x)+(-0.246036407108+
0.240798462224j)*numpy.exp(0.0298913043478j*x)+(-0.175279918721+0.255500269622j)*numpy.exp(0.03125j*x)+(-0.1068959771
22-0.0193382571159j)*numpy.exp(0.0326086956522j*x)+(0.179287240967+0.0884711698517j)*numpy.exp(0.0339673913043j*x)+
(-0.0727410577482-0.327913795246j)*numpy.exp(0.0353260869565j*x)+(0.0594506899557-0.241614151812j)*numpy.exp(0.036684
7826087j*x)+(0.19310098427+0.151369795795j)*numpy.exp(0.0380434782609j*x)+(-0.234993893334+0.0313281826602j)*numpy.e
xp(0.039402173913j*x)+(-0.365722068985+0.0285545452021j)*numpy.exp(0.0407608695652j*x)+(-0.470760261804-0.1384699537
37j)*numpy.exp(0.0421195652174j*x)+(-0.289941859819-0.034611296911j)*numpy.exp(0.0434782608696j*x)+(0.0071672395830+
0.306363818756j)*numpy.exp(0.0448369565217j*x)+(0.111570601653-0.476449285374j)*numpy.exp(0.0461956521739j*x)+(-0.064
5311911822+0.151879156556j)*numpy.exp(0.0475543478261j*x)+(0.0789154712737+0.123058886322j)*numpy.exp(0.0489130434783
j*x)+(0.308350822225-0.321986251638j)*numpy.exp(0.0502717391304j*x)+(-0.205330549438+0.060822552661j)*numpy.exp(0.051
6304347826j*x)+(-0.111432762432-0.196002257036j)*numpy.exp(0.0529891304348j*x)+(-0.186147288585+0.0571708417067j)*num
py.exp(0.054347826087j*x)+(-0.3143266197-0.105088205018j)*numpy.exp(0.0557065217391j*x)+(0.19000154905+0.11156075631
8j)*numpy.exp(0.0570652173913j*x)+(-0.138072497087+0.112684426125j)*numpy.exp(0.0584239130435j*x)+(-0.057339368746+0.
212359713763j)*numpy.exp(0.0597826086957j*x)+(-0.233019181104+0.341870946377j)*numpy.exp(0.0611413043478j*x)+(0.28190
923794-0.443785641915j)*numpy.exp(0.0625j*x)+(-0.0952119988175-0.0444826831503j)*numpy.exp(0.0638586956522j*x)+(-0.24
5611700998-0.0661981914326j)*numpy.exp(0.0652173913043j*x)+(0.175441186289+0.0283829135648j)*numpy.exp(0.066576086956
5j*x)
```

ORIGINAL GRAPH (MAGNIFIED VERSION)



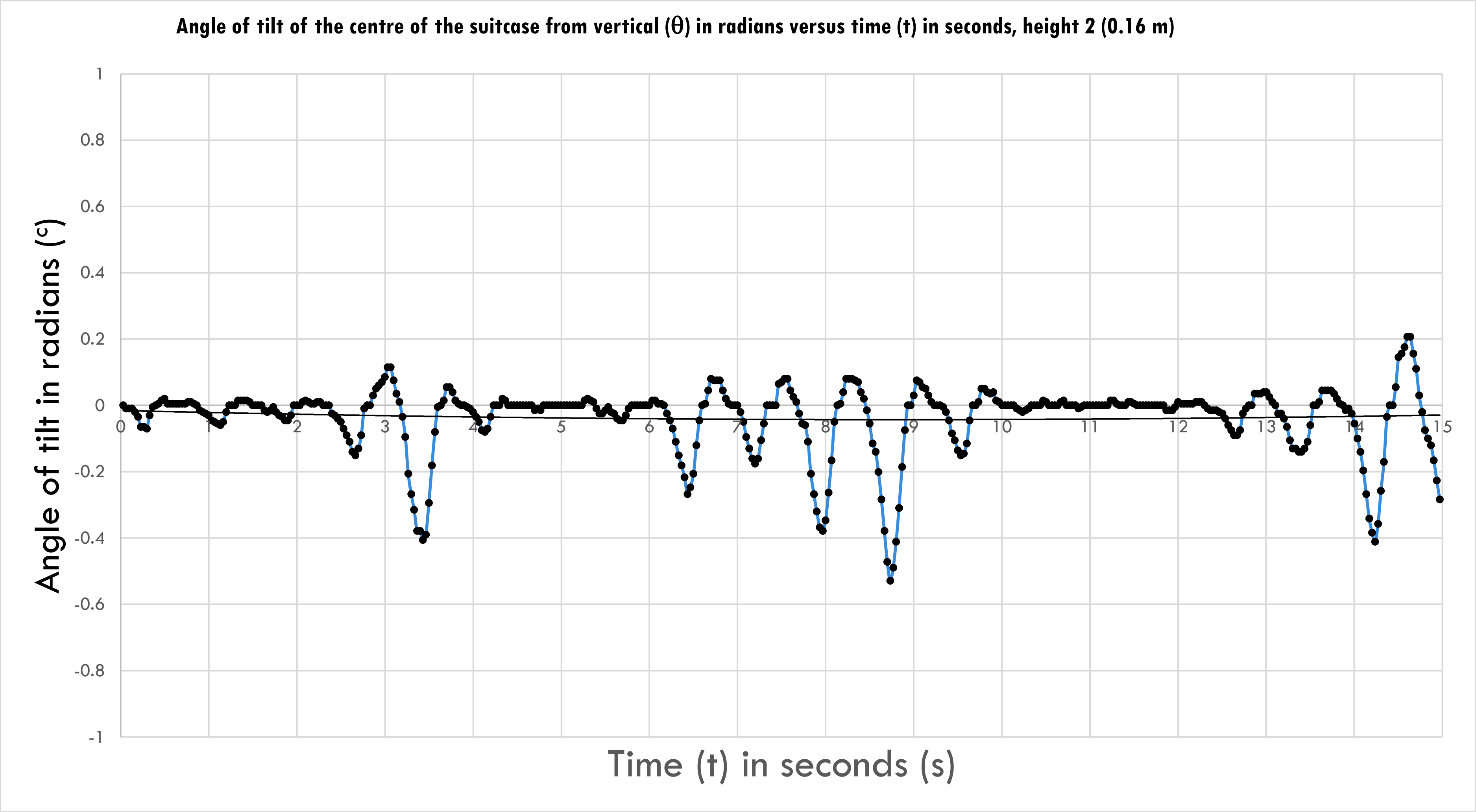
- Angular Displacement vs.Time
- Average range: -0.04 ~ 0.01 (rad)

SECOND DERIVATIVE (ANGULAR ACCELERATION)

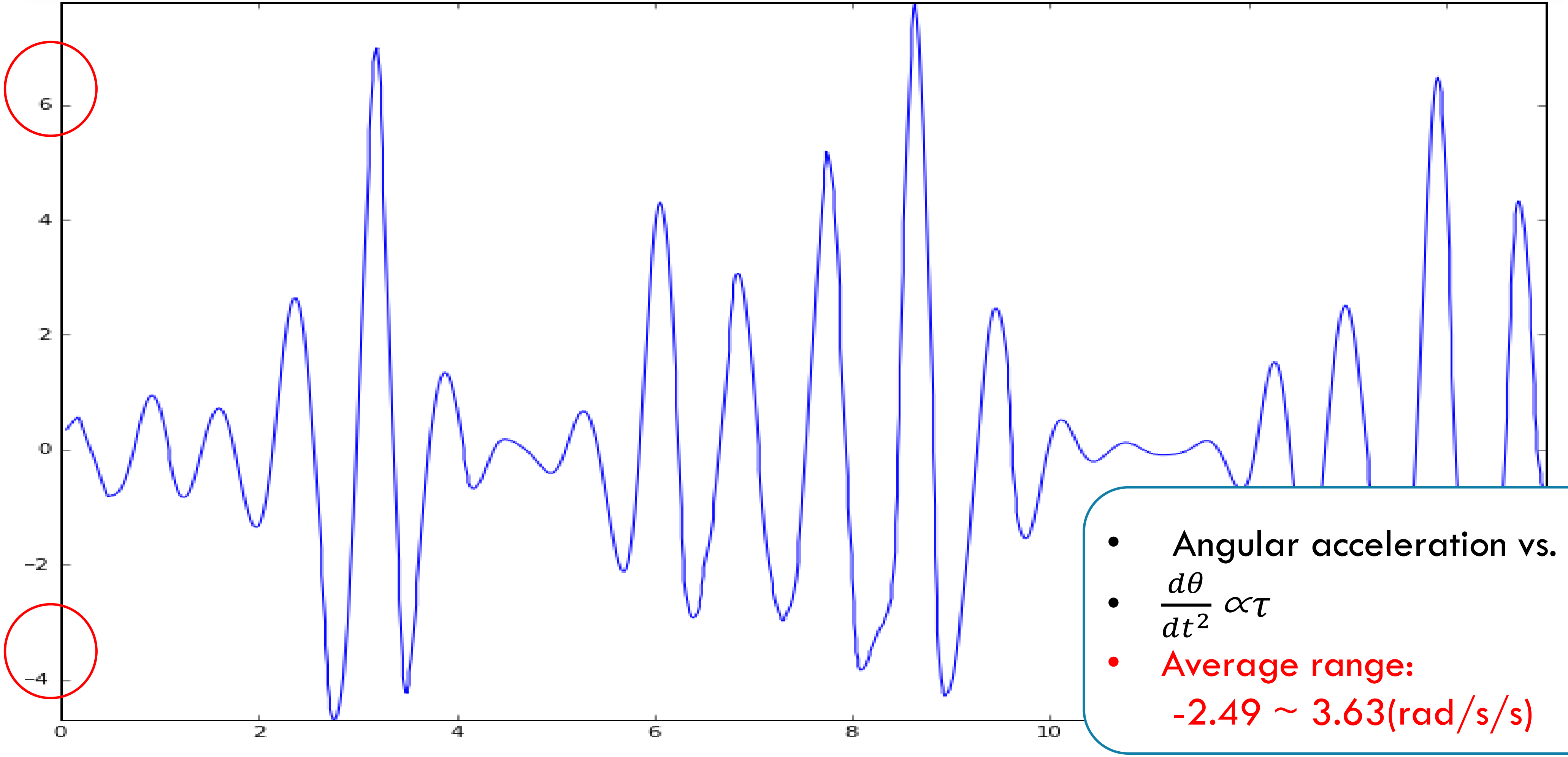


- Angular acceleration vs. time
- $\frac{d\theta}{dt^2} \propto \tau$
- Average range:
  -1.02 ~ 0.88(rad/s/s)

RESULT 2: 0.16M EFFECTIVE HEIGHT OF CM

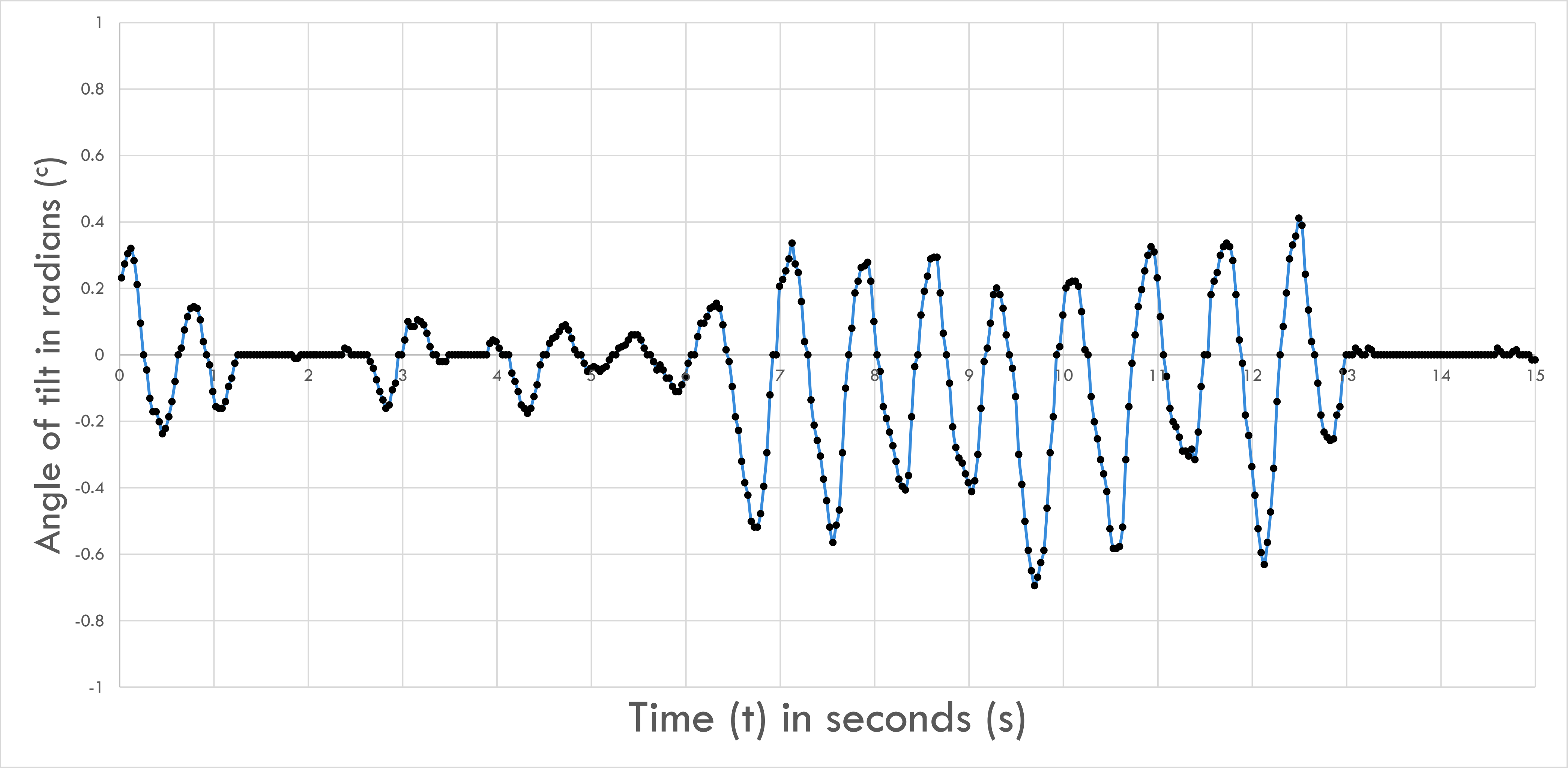


SECOND DERIVATIVE (ANGULAR ACCELERATION)

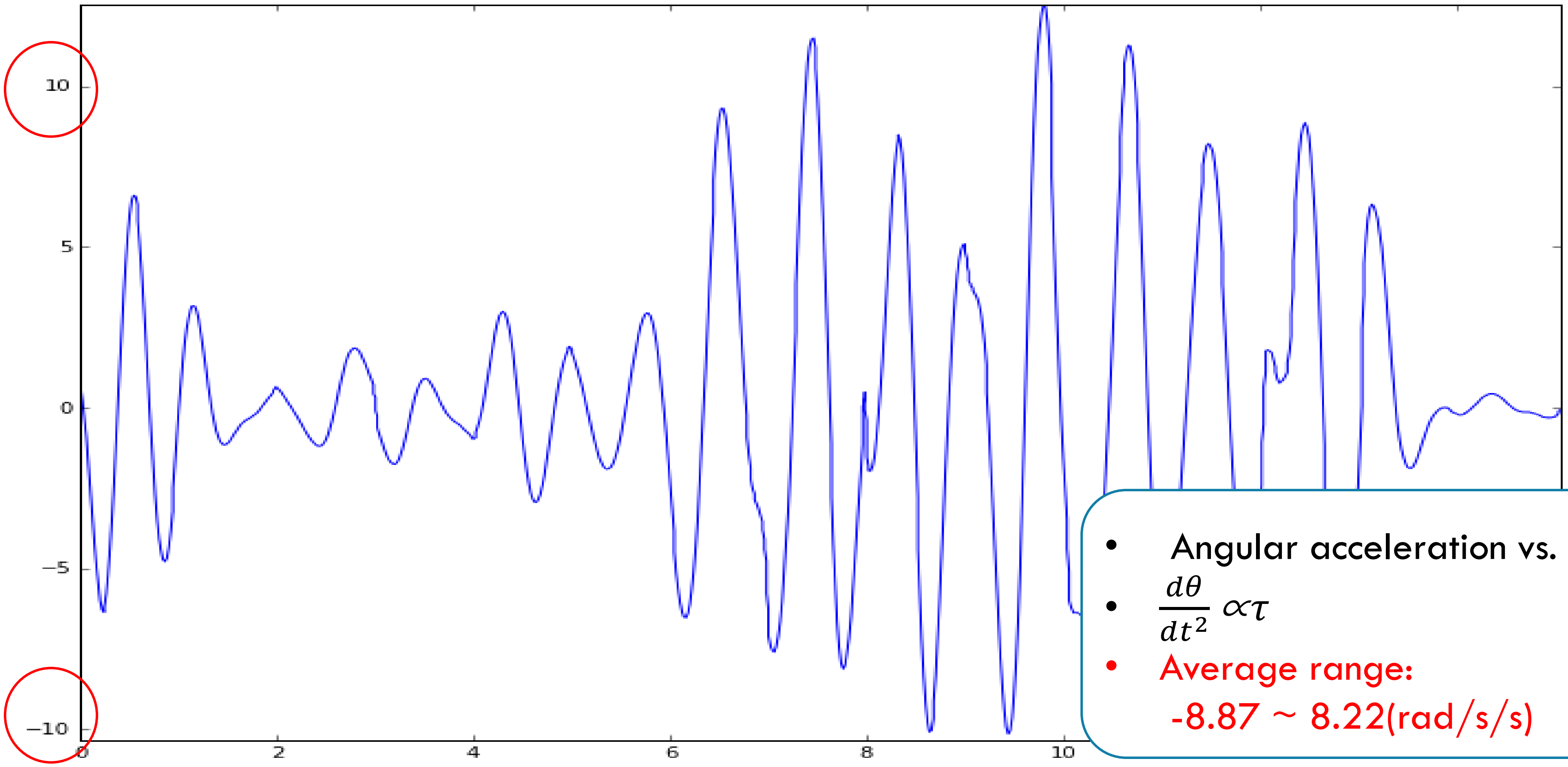


- Angular acceleration vs. time
- $\frac{d\theta}{dt^2} \propto \tau$
- Average range:
  -2.49 ~ 3.63(rad/s/s)

RESULT 3: 0.24M EFFECTIVE HEIGHT OF CM

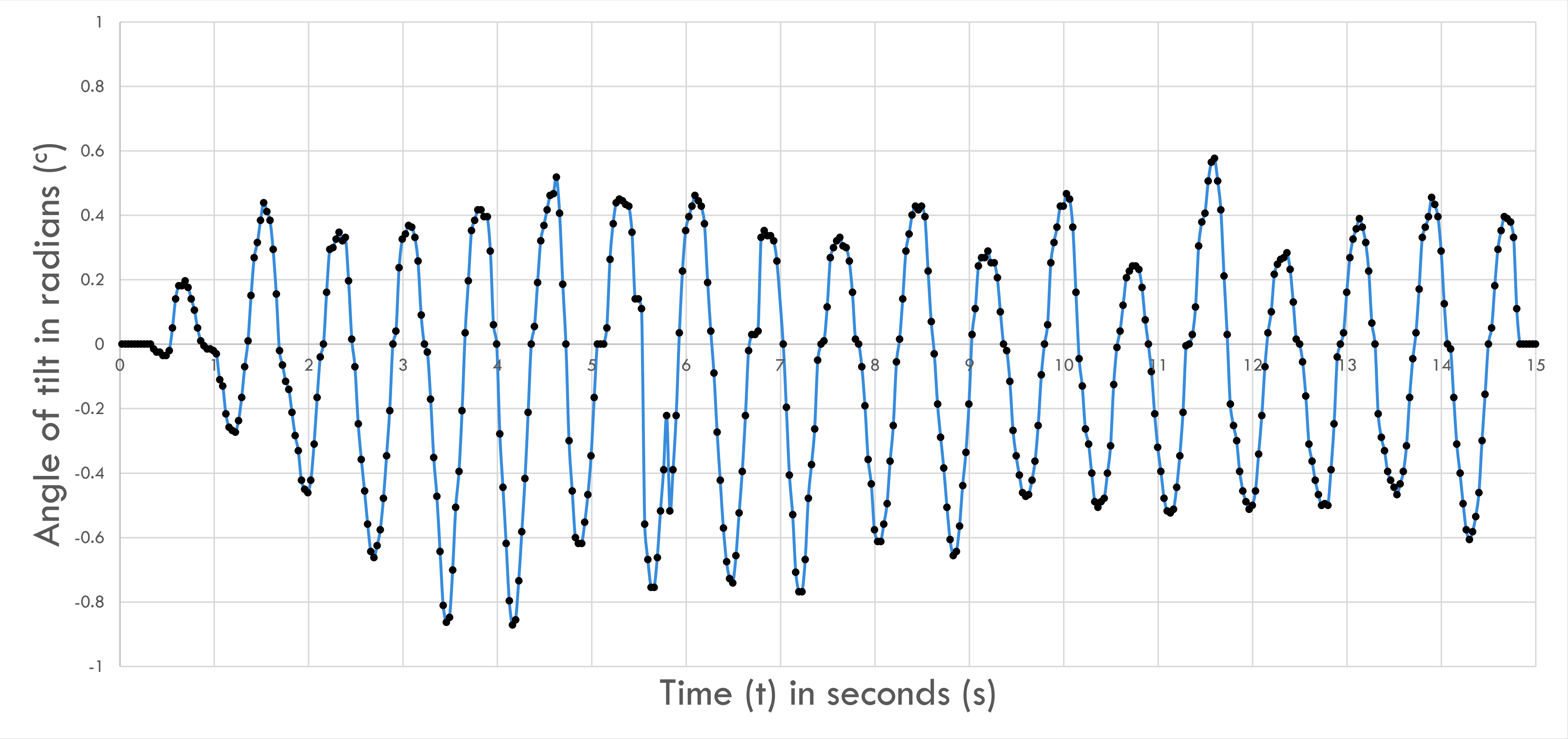


SECOND DERIVATIVE (ANGULAR ACCELERATION)

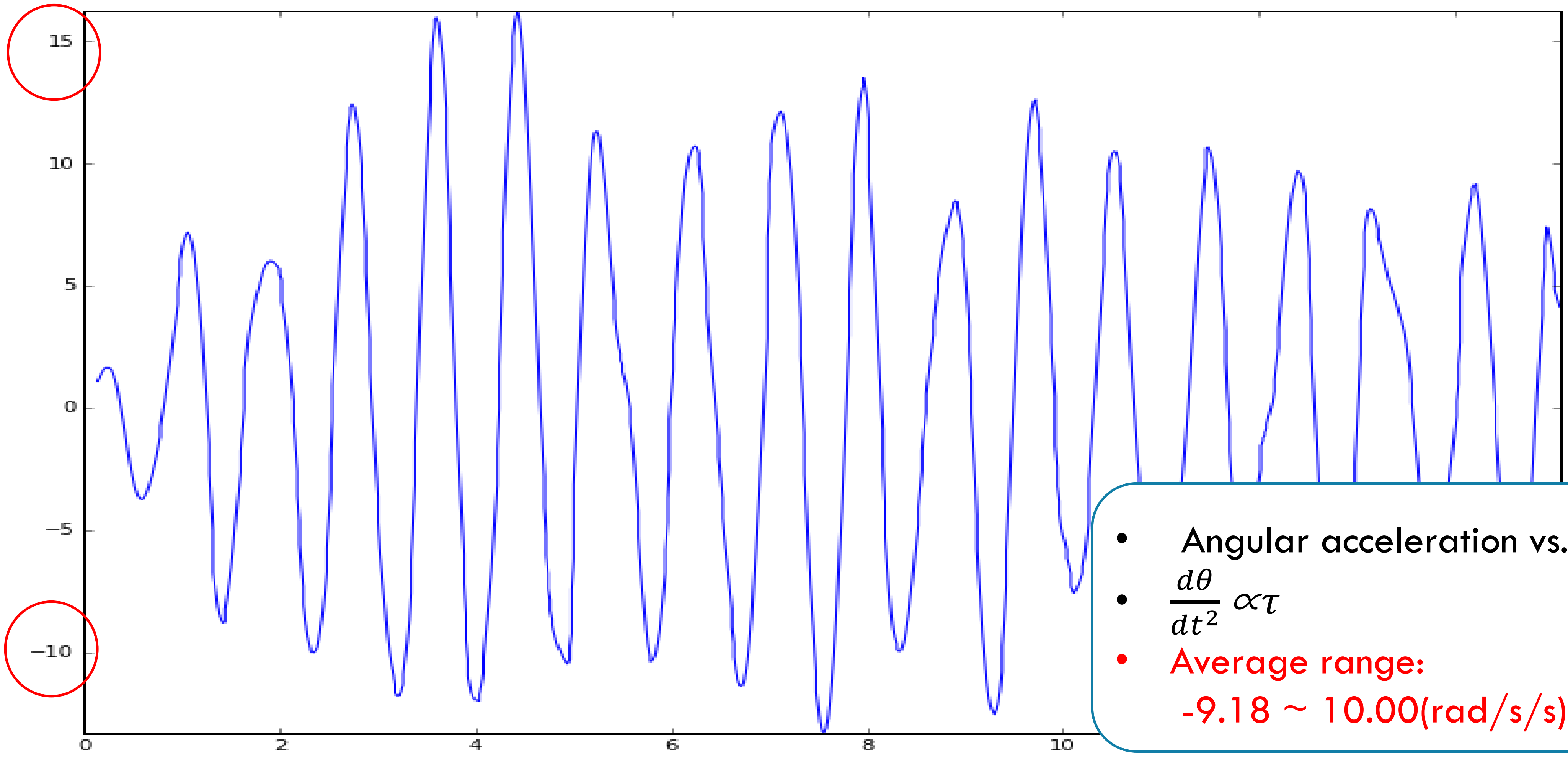


- Angular acceleration vs. time
- $\frac{d\theta}{dt^2} \propto \tau$
- Average range:
  -8.87 ~ 8.22(rad/s/s)

RESULT 4: 0.32M EFFECTIVE HEIGHT OF CM

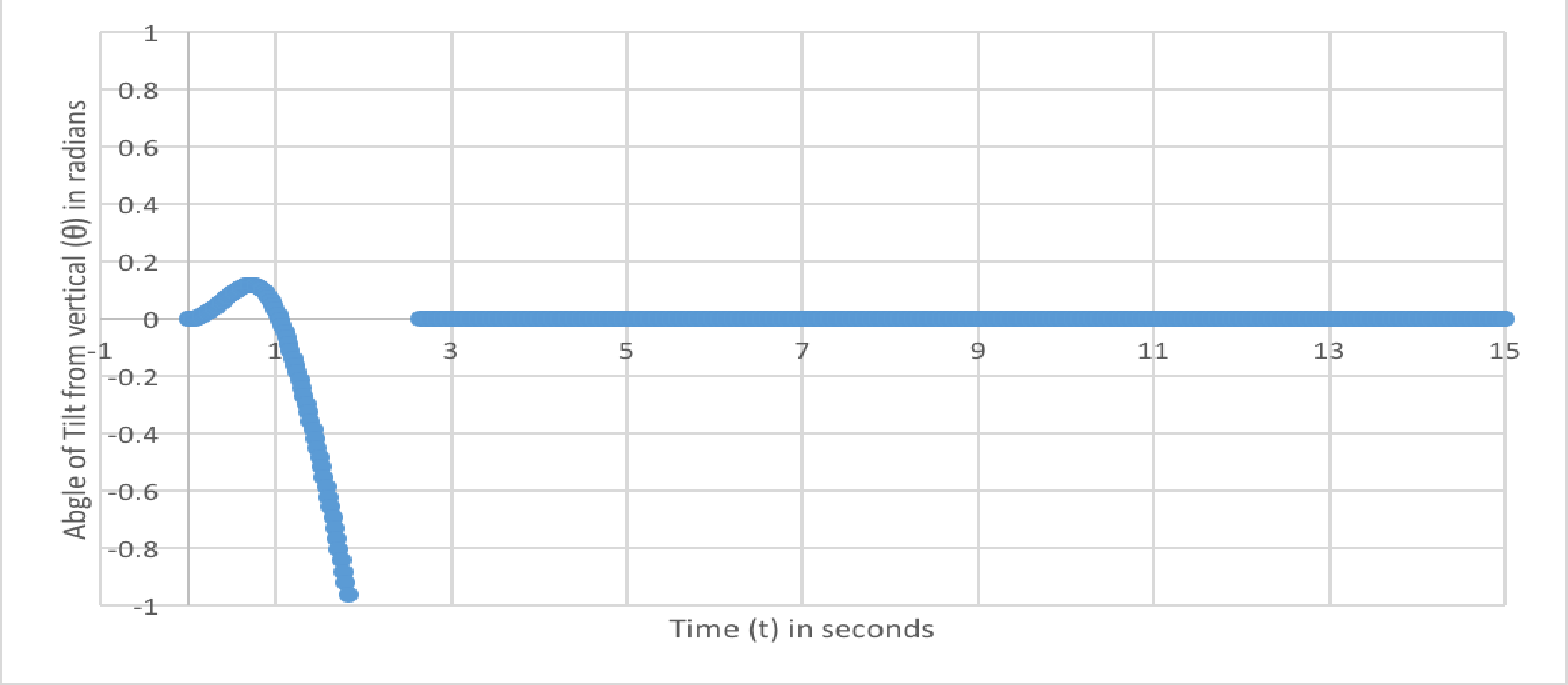


SECOND DERIVATIVE (ANGULAR ACCELERATION)

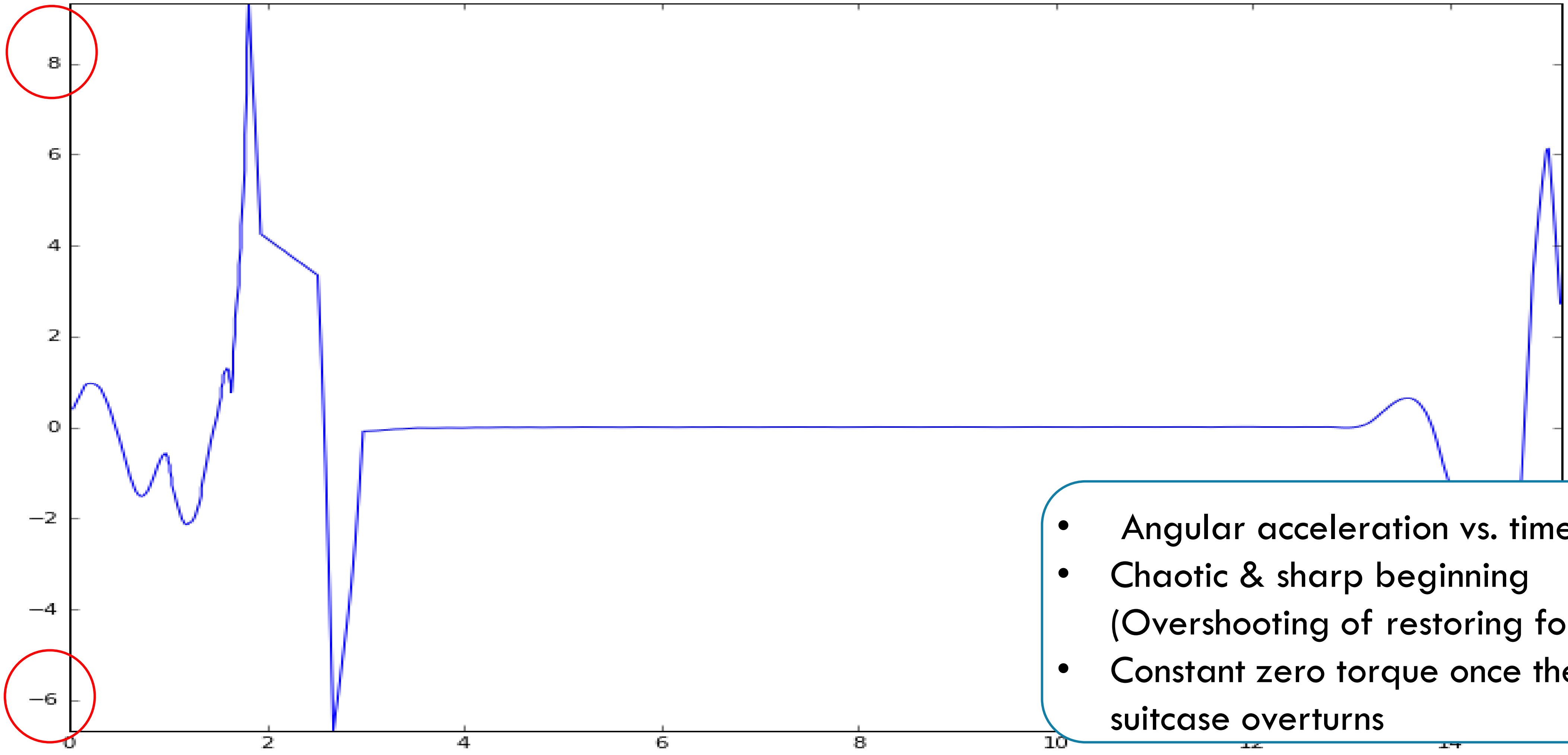


- Angular acceleration vs. time
- $\frac{d\theta}{dt^2} \propto \tau$
- Average range:
  -9.18 ~ 10.00(rad/s/s)

RESULT 5: 0.40M EFFECTIVE HEIGHT OF CM (OVERTURN)



SECOND DERIVATIVE (ANGULAR ACCELERATION)



- Angular acceleration vs. time
- Chaotic & sharp beginning (Overshooting of restoring force)
- Constant zero torque once the suitcase overturns

SUMMARY OF EXPERIMENTAL RESULT (SET A)

Effective Height (m)	Range of angular acceleration (rad/s/s)	Average angular acceleration (rad/s/s)
0.08	-1.02 ~ 0.88	0.95
0.16	-2.49 ~ 3.63	3.06
0.24	-8.87 ~ 8.22	8.54
0.32	-9.18 ~ 10.00	9.59
0.40	Overturn	N/A

As effective height increases:

=Increase in angular acceleration

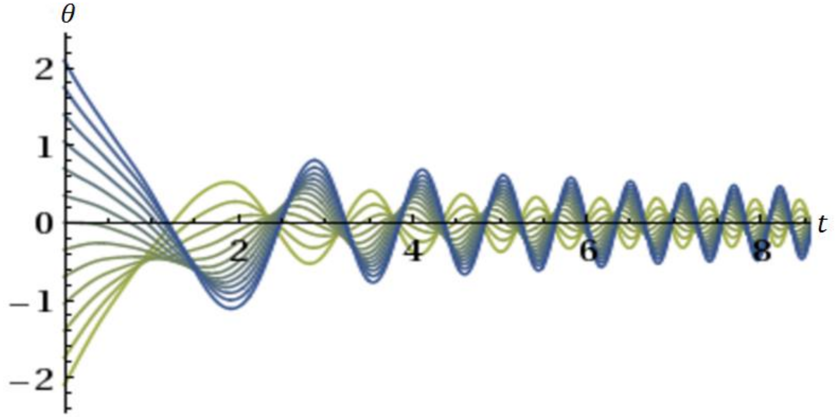
= Increase in torque

= Increase in wobble

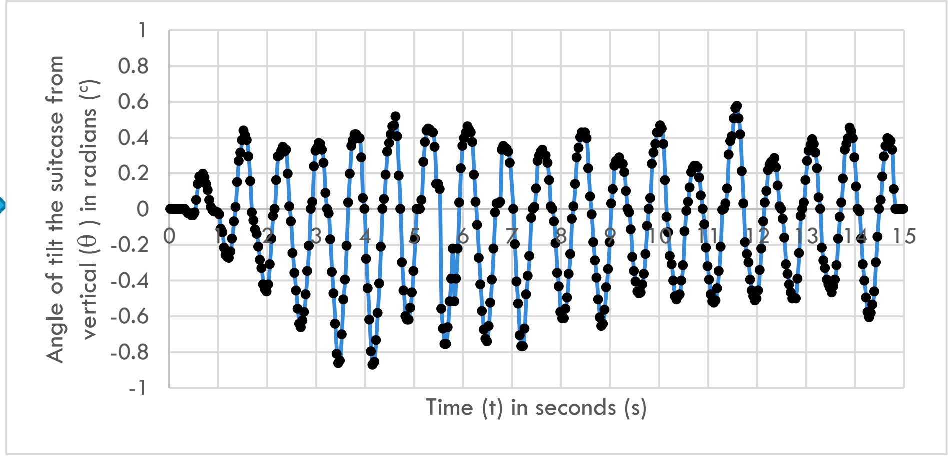
= MORE LIKELY TO OVERTURN

COMPARISON TO THEORETICAL PREDICTION

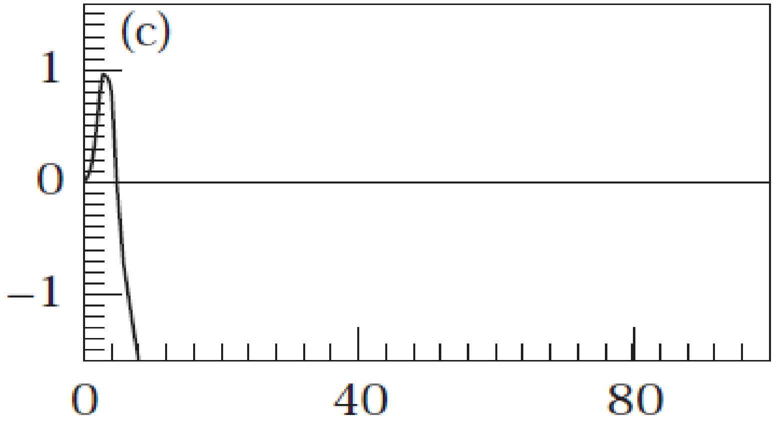
Theory (Oscillation)



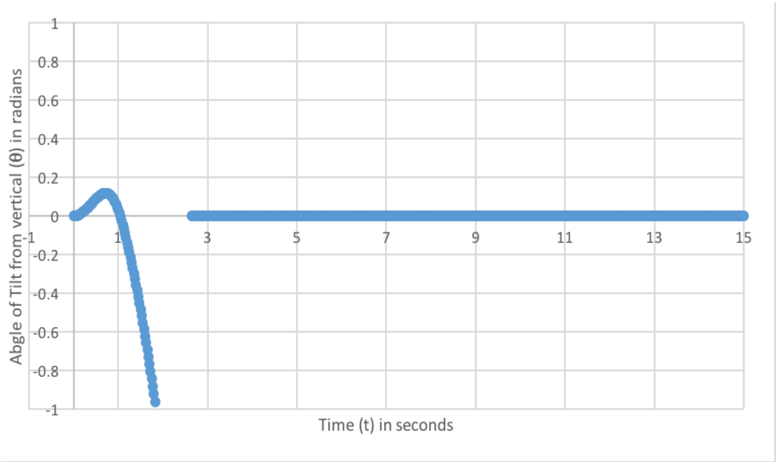
Experiment (Oscillation)



Theory (Overturn)

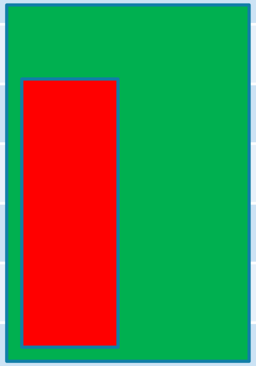


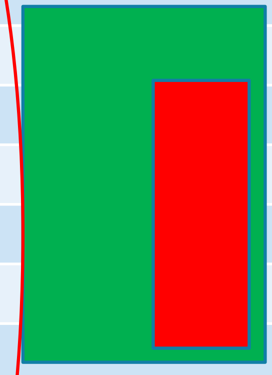
Experiment (Overturn)



Verified!

SUMMARY OF EXPERIMENTAL RESULT (SET B)

Weight Dist.	Speed (m/s)	Angle	Wobbles	Overturn
Pivot wheel	""	70°	1	1
			1	1
			1	1
		60°	1	1
			1	1
			1	1
		50°	5	0
			2	1
			1	1
		40°	1	1
			1	1
			1	1

Weight Dist.	Speed (m/s)	Angle	Wobbles	Overturn
Obstructed wheel	""	70°	0	0
			0	0
			0	0
		60°	0	0
			0	0
			0	0
		50°	0	0
			0	0
			0	0

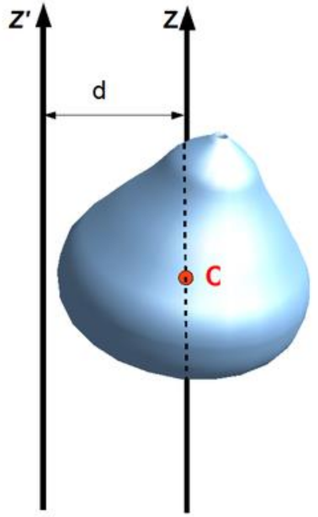
Potential outlier @ 50° for opposite wheel

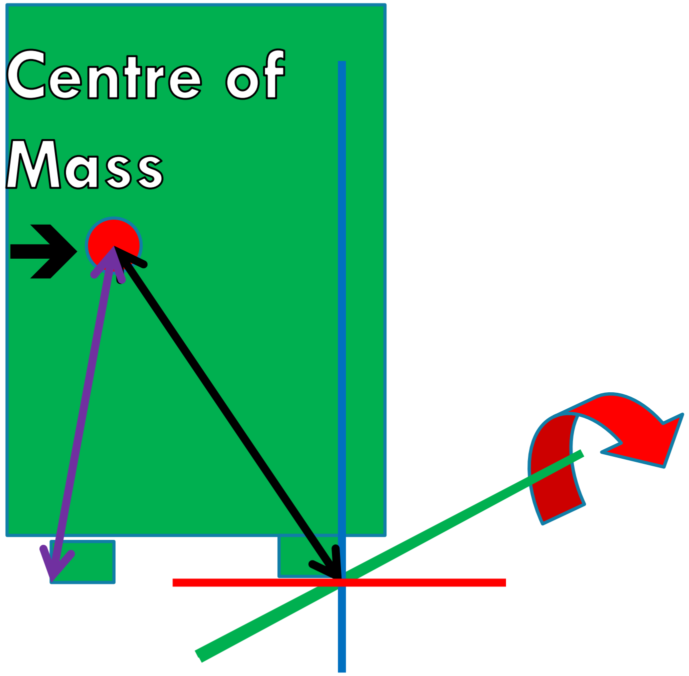
RESULTS DISCUSSION (SET B)

The closer the centre of mass to the obstructed wheel the less wobbles/overturns the suitcase will have (Parallel axis Theorem).

Having the centre of mass closer to the pivot wheel causes more instability.

$$I = I_{CM} + md^2$$





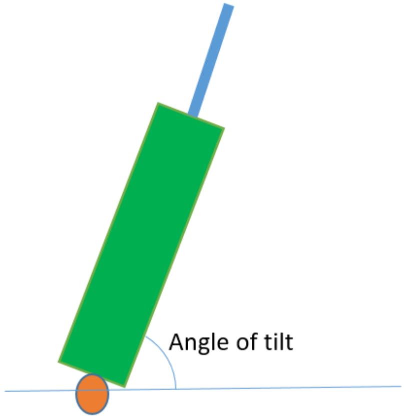
FURTHER CIRCUMSTANCES OF OVERTURN

- Walking speed
- Angle of tilt
- Mass density
- Coefficient of Restitution

1) Angle of tilt vs Walking speed

Degree of tilt	0.81 m/s	0.92 m/s	1.04 m/s
70°	O	O	O
60°	N	O	O
50°	N	O	O
40°	N	N	N

APP FOR WALKING SPEED



ANGLE OF TILT

At 0.81 m/s, stable until high angles.
At 1.04 m/s, stability reduced (high variations in τ_H)

2) LOW WEIGHT:
Mass density vs Walking speed

No added mass	0.92 m/s
70°	O
60°	O
50°	N
40°	N
1.5 kg mass	**0.92 m/s**
70°	O
60°	O
50°	N
40°	N
3 kg mass	**0.92 m/s**
70°	O
60°	N
50°	N
40°	N
30°	N

3) HIGH WEIGHT:
Mass density vs Walking speed

No added mass	0.92 m/s
70°	O
60°	O
50°	O
40°	N
1.5 kg mass	**0.92 m/s**
70°	O
60°	N
50°	N
40°	N
3 kg mass	**0.92 m/s**
70°	N
60°	N
50°	N
40°	N
30°	N

For low weight:
As mass is increased, stability increases.

$\tau_w \downarrow$

For high weight:
As mass is increased, stability decreases.

$\tau_w \uparrow$

4) COR vs Walking speed

Indoor (COR 0.35)	0.81 m/s	0.92 m/s	1.04 m/s	Outdoor (COR 0.68)	0.81 m/s	0.92 m/s	1.04 m/s
70°	O	O	O	70°	O	O	O
60°	N	O	O	60°	O	O	O
50°	N	O	O	50°	O	O	O
40°	N	N	N	40°	O	-	-

- At lower walking frequency, suitcase was less stable for trials outdoors, due to higher COR and less energy loss.
- At higher walking frequency, energy loss is less significant due to balancing effect of higher walking frequency.

CONCLUSION

Investigated circumstances when suitcase wobbles and turns over.

1. Developed mathematical simulation to model the wobble (FFT)
2. Simulation verified -> Comparison to theory
3. Studied effect of varied packing of the luggage
   A. Significance of effective height of CM
   B. Effect of other positions of CM vs. Wheel Position
4. Considered other factors that influence the wobble
   A. Walking Frequency
   B. Angle of tilt
   C. Coefficient of Restitution
   D. Different mass density

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Thanks for your time

Mr Richard Jones

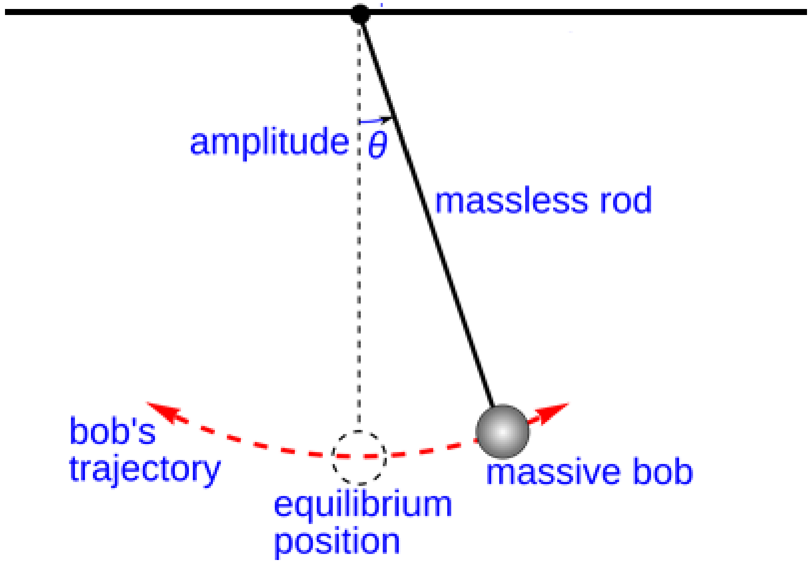
Finn Connolly

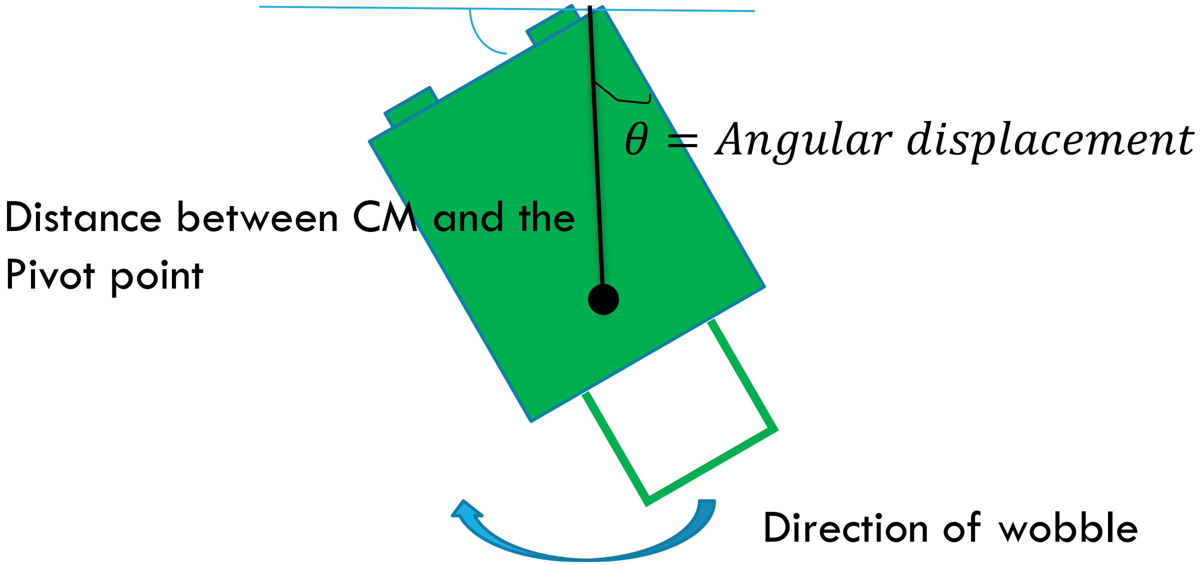
THANK YOU!

Team of Australia

Reporter: Jeong Han Song

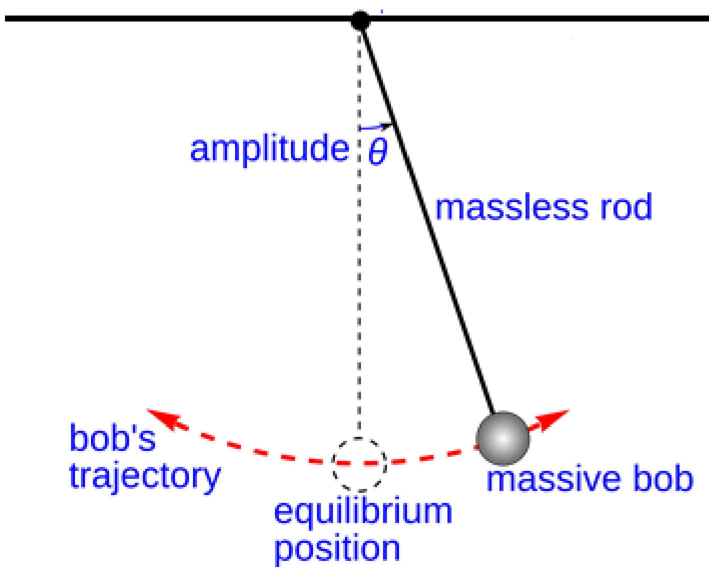
SUITCASE AS AN "INVERTED PENDULUM"

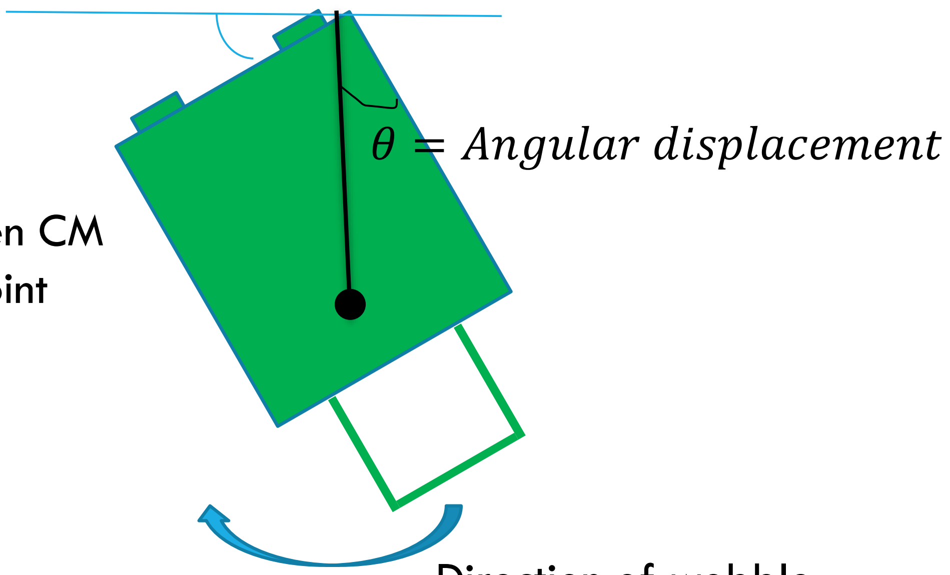




Simple Pendulum	Crazy Suitcase
Massive Bob	Centre of mass of packing
Amplitude θ	Angular Displacement θ
Length of Rod	Distance between CM and the Pivot
Bob's trajectory	Direction of wobble

SUITCASE AS AN "INVERTED PENDULUM"





Distance between CM and the Pivot point

Direction of wobble

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$$

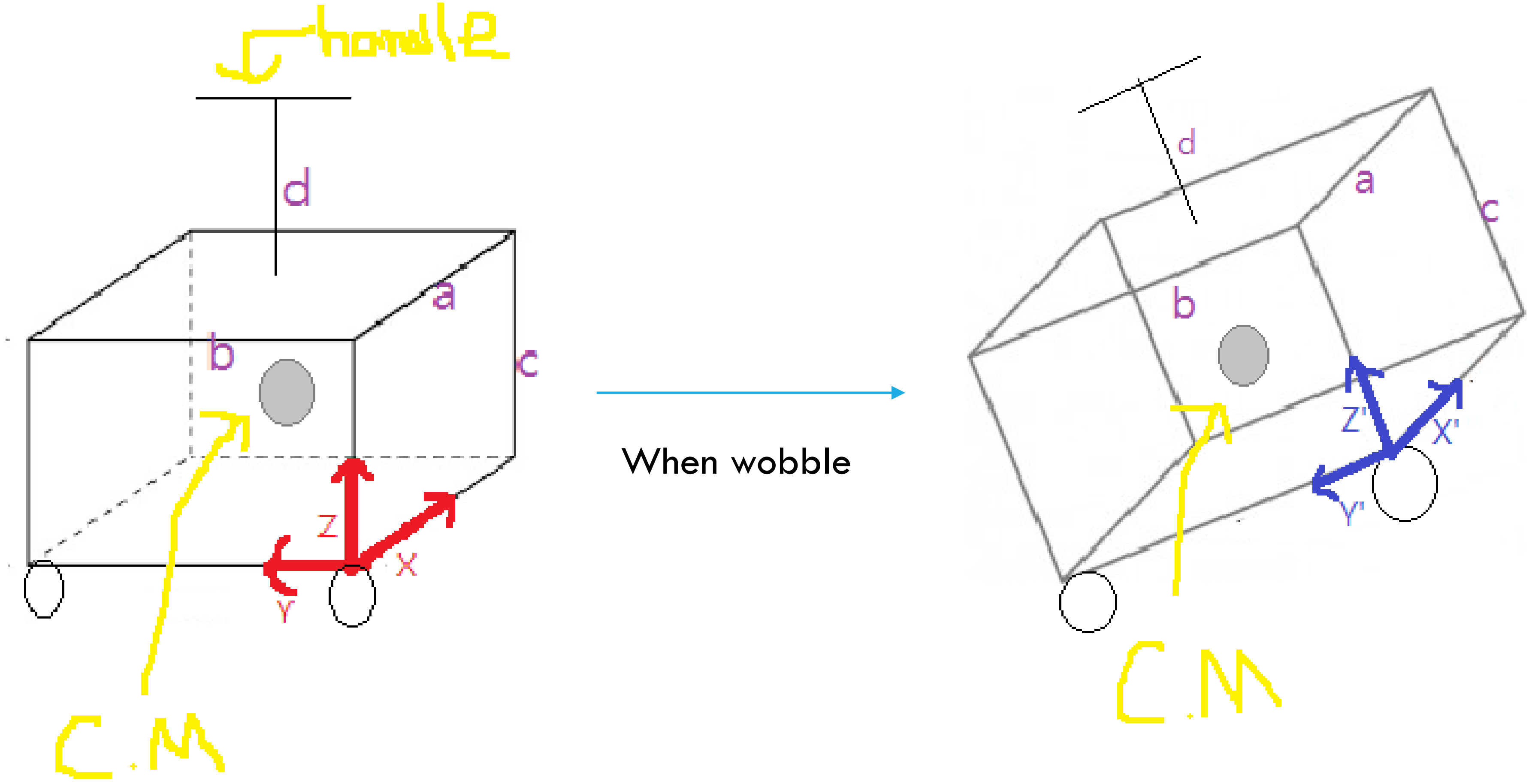
Length of Rod

Relation between effective height and the frequency of oscillation

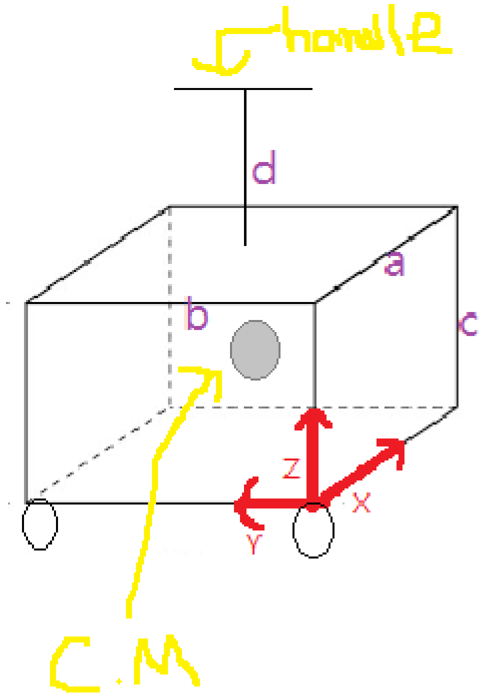
ASSUMPTIONS

1. A constant initial disturbance was employed to start the wobble (e.g. instability) of a suitcase.
2. The suitcase and its wheels were regarded as rigid bodies.
3. Human always responded to the instability by exerting restoring torques that were ***just enough*** to stabilise the motion
4. Human response time was always constant

SUITCASE IN 3-DIMENSION



SUITCASE IN 3-DIMENSION

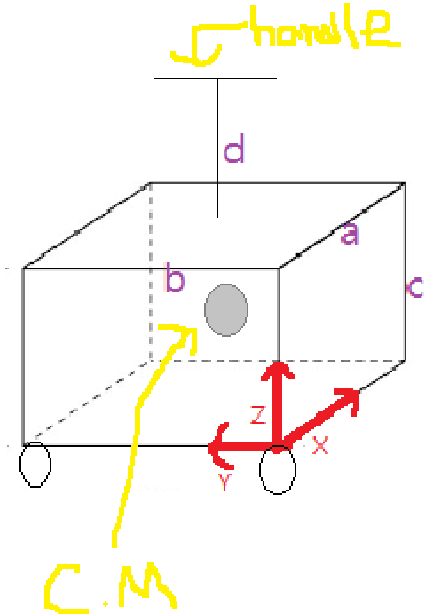


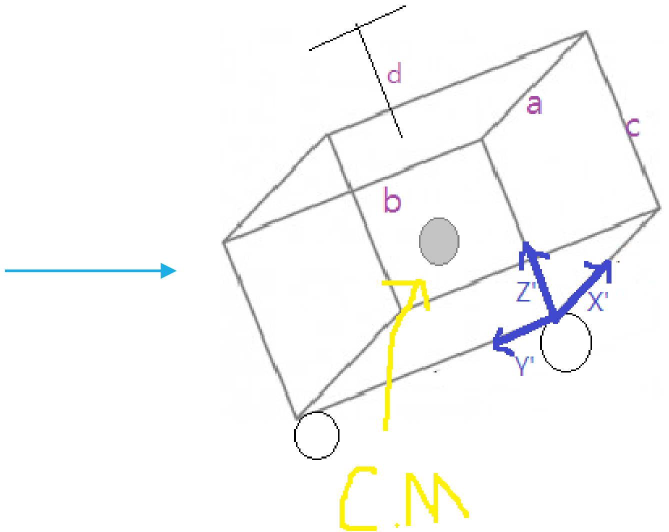
$$\text{CM} \rightarrow \left(\frac{a}{2}, \frac{b}{2}, c\right) = (x, y, z)$$

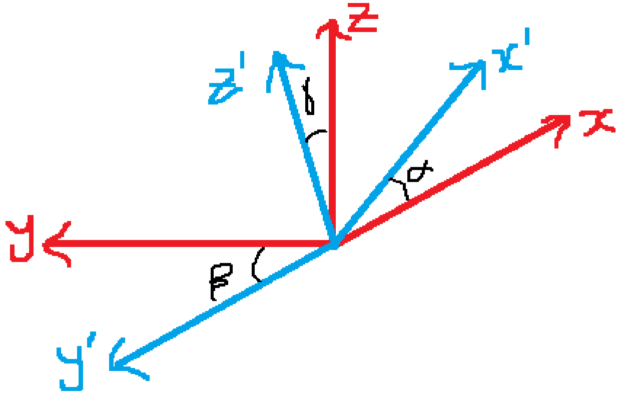
$$\text{Handle} \rightarrow \left(\frac{a}{2}, \frac{b}{2}, c + d\right) = (X, Y, Z)$$

SUITCASE IN 3-DIMENSION

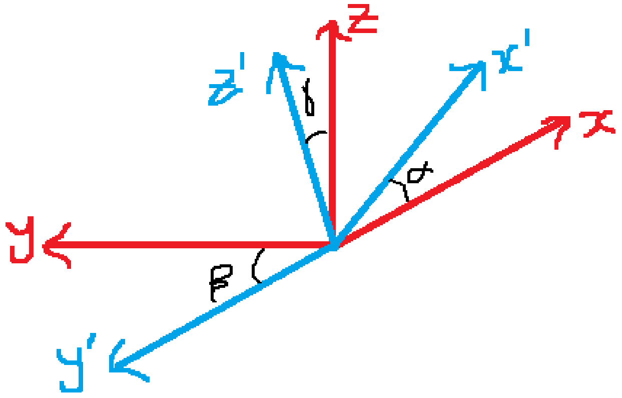
Change in co-ordinate when tilted







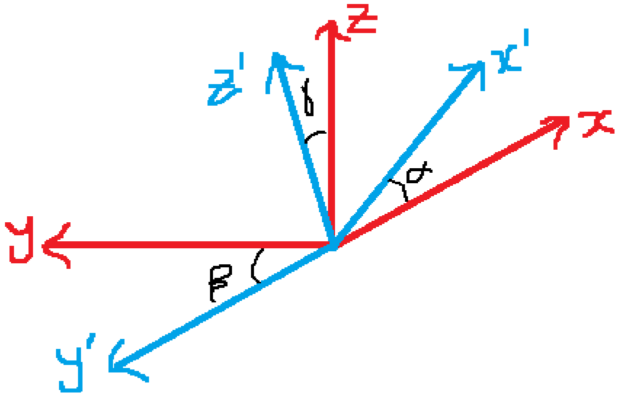
SUITCASE IN 3-DIMENSION



When a suitcase is initially pulled by a person, there will be a rotation about y-axis.

This rotation is followed by x-axis and z-axis.

POSSIBLE WOBBLING MOTI - GENERAL FORMULA



$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix} \begin{pmatrix} \frac{a}{2} \\ \frac{b}{2} \\ c \end{pmatrix}$$

NEW Centre of Mass

z-axis

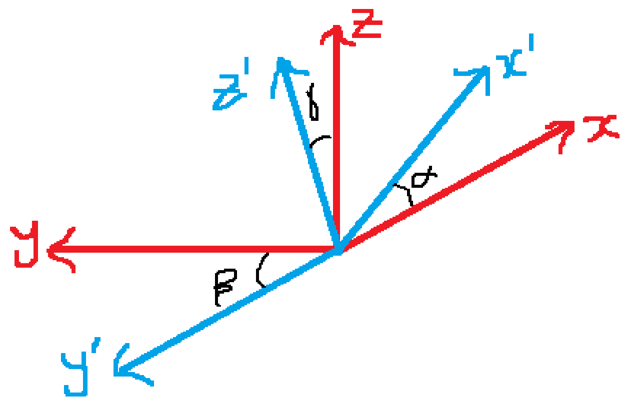
x-axis

y-axis

PREVIOUS Centre of Mass

SHIFT IN CENTRE OF MASS OF AN OBJECT

POSSIBLE WOBBLING MOTI - GENERAL FORMULA



$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix} \begin{pmatrix} \frac{a}{2} \\ \frac{b}{2} \\ d+z \end{pmatrix}$$

NEW Centre of Mass — z-axis — x-axis — y-axis — PREVIOUS Centre of Mass

SHIFT IN CENTRE OF MASS OF A HANDLE

POSSIBLE WOBBLING MOTIONS - GENERAL FORMULA

$$R_z R_x R_y = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix}$$

$$= \begin{pmatrix} \cos a & \sin\alpha\cos\beta & \sin\alpha\sin\beta \\ -\sin a & \cos\alpha\cos\beta & \cos\alpha\sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix}$$

$$= \begin{pmatrix} \cos a\cos\gamma - \sin a\sin\beta\sin\gamma & \sin\alpha\cos\beta & \cos a\sin\gamma + \sin a\sin\beta\cos\gamma \\ -\cos a\sin\beta\sin\gamma - \sin a\cos\gamma & \cos\alpha\cos\beta & \cos a\sin\beta\sin\gamma - \sin a\sin\gamma \\ -\cos\beta\sin\gamma & -\sin\beta & \cos\beta\cos\gamma \end{pmatrix}$$